Thermodynamics and contact geometry
Lecture 2: More differential geometry and distributions

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Summary from yesterday

Yesterday's main topic was vector fields on an n-dimensional manifold M.

- A vector field is a section of the bundle TM, i.e. a map $M \to TM$ taking each $p \in M$ to a vector in T_pM.
- Any set of coordinates $x^1, \dots, x^n$ on M gives rise to a set of n vector fields $\partial_{x^1}, \dots, \partial_{x^n}$ that span T_pM for every point p in the coordinate neighborhood.
- In these coordinates a general vector field takes the form

$$X = a^1(x)\partial_{x^1} + \cdots + a^n(x)\partial_{x^n}.$$

We also discussed distributions on M. A regular distribution of rank r was defined as a collection of linear subspaces $\Pi_p \subset T_pM$ depending smoothly on M. We say that a set of vector fields $X_1, \dots, X_r$ span the distribution if the vectors $X_1|_p, \dots, X_r|_p$ span Π_p at each point p. By Frobenius' theorem, there exist r-dimensional integral manifolds if and only if $[X_i, X_j] \in \langle X_1, \dots, X_r \rangle$ for all i, j.

Differential 1-forms

Let T_p^*M denote the vector space dual to T_pM, i.e. the space of linear functions $T_pM \to \mathbb{R}$. It is a vector space of dimension n. Let $T^*M = \sqcup_{p\in M} T_p^*M$. A section of this bundle is called a 1-form. Let $dx^1|_p, \ldots, dx^n|_p$ denote the dual basis to $\partial_{x^1}|_p, \cdots, \partial_{x^n}|_p$ at each point p:

$$dx^i|_p(\partial_{x^j}|_p) = \delta^i_j.$$

We denote by dx^i the section given by $p \mapsto dx^i|_p$. A general 1-form can be written in coordinates as

$$\alpha = b_1(x)dx^1 + \cdots + b_n(x)dx^n.$$

For $X = a^1(x)\partial_{x^1} + \cdots + a^n(x)\partial_{x^n}$, we have

$$\alpha(X) = b_1(x)a^1(x) + \cdots b_n(x)a^n(x).$$

Another way to specify a distribution

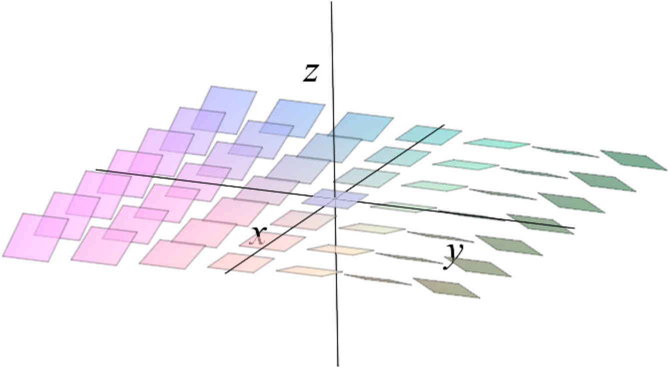
Recall that a regular distribution or rank r on M was defined as a collection of r-dimensional subspaces $\Pi_p \subset T_pM$ depending smoothly on the point $p \in M$. We know from linear algebra that any linear subspace in T_pM can be given as the kernel to a set of linear functions. This means that we can, instead of giving the vector fields $X_1, \dots, X_r$ that span Π, give a set of $n-r$ independent 1-forms $\alpha_1, \dots, \alpha_{n-r}$ such that $\alpha_i(X_j) = 0$ for each i, j.

Example: Consider the distribution $\Pi = \langle \partial_x, \partial_y \rangle$ on $\mathbb{R}^3(x, y, z)$. For $\alpha = adx + bdy + cdz$, we have $\alpha(\partial_x) = a$ and $\alpha(\partial_y) = b$. Thus we can write $\Pi = \ker(dz)$.

Exercise: Show that the distribution $\langle x\partial_y - y\partial_x, \partial_z \rangle$ can be given as the kernel of the 1-form $xdx + ydy$. Note that this is true only outside the line given by $x = y = 0$ (the z-axis).

The contact distribution

Consider the distribution $\mathcal{C} = \langle \partial_x + y\partial_z, \partial_y \rangle$ on $\mathbb{R}^3(x, y, z)$.



It is easy to see that both $\partial_x + y\partial_z$ and ∂_y are in the kernel of $dz - ydx$. Thus $\mathcal{C} = \ker(dz - ydx)$.

The contact distribution on $\mathbb{R}^{2n+1}$

Let $x^1, \ldots, x^n, u, p_1, \ldots, p_n$ be coordinates on $\mathbb{R}^{2n+1}$. The contact distribution $\mathcal{C}$ on $\mathbb{R}^{2n+1}$ can be given as the kernel of the one-form

$$\theta = du - p_1 dx^1 - \cdots - p_n dx^n = du - p_i dx^i.$$

The rank of $\mathcal{C} = \ker(\theta)$ is $2n$. Let $D_{x^i} = \partial_{x^i} + p_i \partial_u$. The distribution $\mathcal{C}$ is spanned by

$$D_{x^1}, \ldots, D_{x^n}, \partial_{p_1}, \ldots, \partial_{p_n}$$

since these are independent and $\theta(D_{x^i}) = 0$, $\theta(\partial_{p_n}) = 0$ (verify this). Let us compute the Lie brackets.

$$[D_{x^i}, D_{x^j}] = 0, \qquad [\partial_{p_i}, \partial_{p_j}] = 0, \qquad [\partial_{p_i}, D_{x^j}] = \delta^i_j \partial_u.$$

The equalities $[\partial_{p_i} D_{x^i}] = \partial_u$ show that $\mathcal{C}$ is not integrable, meaning that there are no $2n$-dimensional integral manifolds.

The algebra of differential forms

It is possible to check integrability of distributions by looking at the 1-forms defining them. In order to do this, we need to extend the space of 1-forms.

Definition

A k-form ω on M is a collection of multilinear and skew-symmetric maps $\omega_p \colon T_pM \times \cdots \times T_pM \to \mathbb{R}$ depending smoothly on p.

We call functions on M 0-forms and denote the space of k forms by $\Omega^k(M)$ for $k = 0, \ldots, n$. Let $\Omega(M) = \oplus_{k=0}^n \Omega^k(M)$. For k vector fields $X_1, \ldots, X_k$ we let $\omega(X_1, \ldots, X_n) \in C^\infty(M)$ be the function defined by $p \mapsto \omega_p(X_1|_p, \ldots, X_k|_p)$.

There exists a product on $\Omega(M)$, called the wedge product and denoted by $\wedge$. For $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, we have

$$\alpha \wedge \beta(X_1, \ldots, X_{k+l}) = \sum_{\sigma \in \mathsf{Sh}_{k,l}} (-1)^\sigma \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})$$

where $\mathsf{Sh}_{k,l} \subset S_{k+l}$ are (k,l)-shuffles, i.e. permutations satisfying $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

In coordinates

The wedge product on 1-forms is skew-symmetric. In particular $dx^i \wedge dx^j = -dx^j \wedge dx^i$ and $dx^i \wedge dx^i = 0$. The product of two 1-forms

$$\alpha = a_i dx^i, \qquad \beta = b_i dx^i$$

is the 2-form

$$\alpha \wedge \beta = a_1 b_1 dx^1 \wedge dx^1 + a_1 b_2 dx^1 \wedge dx^2 + \cdots + \cdots + a_n b_{n-1} dx^n \wedge dx^{n-1} + a_n b_n dx^n \wedge dx^n$$
$$= (a_1 b_2 - a_2 b_1) dx^1 \wedge dx^2 + (a_1 b_3 - a_3 b_1) dx^1 \wedge dx^3 + \cdots + (a_{n-1} b_n - a_n b_{n-1}) dx^{n-1} \wedge dx^n.$$

In the same way that 1-forms can be written in a $C^\infty(M)$-linear combination of $dx^1, \ldots, dx^n$, a 2-form can be written in a $C^\infty(M)$-linear combination of $dx^i \wedge dx^j$ where $1 \leq i < j \leq n$.

Exercise: Show that the 2-form $dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ on $\mathbb{R}^4$ is not the product of two 1-forms.

In coordinates

More generally, a k-form can be written as a $C^\infty(M)$-linear combination of the k-forms $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$:

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \cdots i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Notice that there are up to $\binom{n}{k}$ nonzero terms in this sum and, in particular, when $k = n$ there is only one term.

For $f \in C^\infty(M) = \Omega^0(M)$ and $\alpha \in \Omega^k(M)$ we have $f \wedge \alpha = f\alpha$.

The exterior derivative

There exists a linear operator d on $\Omega(M)$ which takes k-forms to $(k+1)$-forms: $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$. It is defined uniquely by the conditions

- $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$ for $\alpha \in \Omega^k(M)$,
- $d(d\alpha) = 0$,
- $df = \frac{\partial f}{\partial x^i} dx^i$ in local coordinates.

For a vector field X, we have $df(X) = X(f)$.

For k-forms we have

$$d\Big(\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \Big) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} da_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The importance of d comes from the fact that it is independent of the coordinate system that is used.

Another formulation of Frobenius' theorem

For a set $\{\omega_1, \dots, \omega_q\}$ of 1-forms on M, let

$$\ker_p(\omega_1, \dots, \omega_q) = \{v \in T_pM \mid \omega_i(v) = 0\ \forall i\} \subset T_pM$$

and let $\ker(\omega_1, \dots, \omega_q) = \sqcup_{p \in M} \ker_p(\omega_1, \dots, \omega_q) \subset TM$ be the corresponding subbundle. If $\dim \ker_p(\omega_1, \dots, \omega_q)$ is the same for each $p \in M$, then the distribution Π defined by $\Pi_p = \ker_p(\omega_1, \dots, \omega_q)$ is a regular distribution.

Theorem

Let $\Pi = \ker(\omega_1, \dots, \omega_q)$ be a distribution of the type described above. Then Π is integrable if and only if $d\omega_i|_\Pi = 0$ for $i = 1, ..., q$.

In other words Π is integrable if and only if for each pair X, Y on which every ω_i vanishes we have $d\omega_i(X, Y) = 0$ for every i.

Frobenius' theorem for distributions of hyperplanes

There is a special case when $\Pi = \ker(\omega)$ for a single one-form with $\omega|_p \neq 0$ for every $p \in M$. Then $\Pi_p \subset T_pM$ is an $(n-1)$-dimensional hyperplane. In this case Frobenius' theorem says that Π is integrable if and only if $d\omega|_\Pi = 0$.

Theorem

The condition $d\omega|_\Pi = 0$ is equivalent to $\omega \wedge d\omega = 0$.

Proof.

$\omega \wedge d\omega = 0$ if and only if $\omega \wedge d\omega(X, Y, Z) = 0$ for any three vectors X, Y, Z. It is sufficient to verify the statement for X, Y, Z satisfying $\omega(X) = \omega(Y) = 0$ and $\omega(Z) \neq 0$*. In that case we have

$$\omega \wedge d\omega(X, Y, Z) = \omega(Z) d\omega(X, Y).$$

□

*Note that if, for example, both $\omega(X) \neq 0$ and $\omega(Z) \neq 0$, then $\omega \wedge d\omega(X, Y, Z) = \omega \wedge d\omega(X, Y, Z) + \omega \wedge d\omega(aZ, Y, Z) = \omega \wedge d\omega(X + aZ, Y, Z)$, and you can always choose a so that $\omega(X + aZ) = 0$.

Contact structures

Definition

A regular distribution $\Pi = \ker(\omega)$ on a $(2n+1)$-dimensional manifold M is called a contact structure on M if $\omega \wedge (d\omega)^n \neq 0$. If such a structure is provided on M, we call M a contact manifold.

For a regular distribution $\Pi = \ker(\omega)$, we noticed that 2-form $d\omega|_\Pi$ can be interpreted to measure the nonintegrability of Π. In particular, if $d\omega|_\Pi = 0$, then Π is integrable. The other extreme case is when the rank of $d\omega|_\Pi$ is equal to the rank of Π. This means that for any nonzero vector field X there exists a vector field Y such that $\omega(X, Y) \neq 0$, something that can only happen if the rank of Π is even (so M must be odd). On a $(2n+1)$-dimensional manifold the condition that $d\omega|_\Pi$ is of rank $2n$ is equivalent to $\omega \wedge (d\omega)^n \neq 0$. In this sense, contact distributions are maximally far from being integrable.

Since contact structures are not integrable, they have no $(2n+1)$-dimensional integral manifolds. Instead, their integral manifolds have at most dimension n.

The main example

Consider $\mathbb{R}^{2n+1}$ with the distribution $\Pi = \ker(du - p_i dx^i)$. We already showed by looking at the vector fields generating Π that it is not integrable. Let us now do it by looking at the 1-form $\omega = du - p_i dx^i$. We have

$$d\omega = 0 - dp_i \wedge dx^i - p_i \wedge ddx^i = dx^i \wedge dp_i.$$

Further, we have

$$\omega \wedge d\omega = (du - p_i dx^i) \wedge (dx^j \wedge dp_j) = du \wedge dx^j \wedge dp_j$$

which is not zero. The distribution Π is therefore not integrable. Since

$$(d\omega)^n = n! dx^1 \wedge dp_1 \wedge \cdots \wedge dx^n \wedge dp_n$$

we have

$$\omega \wedge (d\omega)^n = (du - p_i dx^i) \wedge (d\omega)^n = n! du \wedge dx^1 \wedge dp_1 \wedge \cdots \wedge dx^n \wedge dp_n \neq 0.$$

This means that Π is a contact structure on M.

Integral manifolds of $\ker(du - p_i dx^i)$.

Let us find integral manifolds of dimension n. Let L be an integral manifold and assume that we can take $x^1, ..., x^n$ as coordinates on L. Then L is given by

$$u = f(x), \qquad p_1 = g_1(x), \qquad \ldots, \qquad p_n = g_n(x).$$

We have

$$(du - p_i dx^i)|_L = df(x) - g_i(x)dx = \left(\frac{\partial f}{\partial x^i}(x) - g_i(x)\right) dx^i.$$

The manifold L is an integral manifold if and only if $(du - p_i dx^i)|_L = 0$. In that case all tangent vectors to L will be in $\ker(du - p_i dx^i)$. Thus L is an integral manifold if and only if $g_i(x) = \frac{\partial f}{\partial x^i}(x)$ for $i = 1, ..., n$.

Integral manifolds of dimension n on a $(2n+1)$-dimensional contact manifold are called Legendrian submanifolds.

Example: The jet space

For a function $f \in C^\infty(\mathbb{R}^n)$, let $[f]^1_a = f(a) + \frac{\partial f}{\partial x^i}(a)(x^i - a^i)$ denote its degree 1 Taylor polynomial at $a \in \mathbb{R}^n$ (its 1-jet at a). Define

$$J^1_a(\mathbb{R}^n) = \{[f]^1_a \mid f \in C^\infty(\mathbb{R}^n).$$

Since each 1-jet is determined by $n+1$ numbers $(f(a), \frac{\partial f}{\partial x^i}(a))$ we see that $\dim J^1_a(\mathbb{R}^n) = n+1$. Let now $J^1(\mathbb{R}^n) = \sqcup_{a \in \mathbb{R}} J^1_a(\mathbb{R}^n)$ denote the $(2n+1)$-dimensional space of all 1-jets (at any point). We may introduce coordinates x^i, u, p_i on $J^1(\mathbb{R})$:

$$x^i([f]^1_a) = a^i, \qquad u([f]^1_a) = f(a), \qquad p_i([f]^1_a) = \frac{\partial f}{\partial x^i}(a).$$

This space is a bundle $\pi_1 \colon J^1(\mathbb{R}^n) \to \mathbb{R}^n$ with projection given by $\pi_1([f]^1_a) = a$.

Example: The jet space

Any function $f \in C^\infty(M)$ gives rise to a section $j^1 f$ of π_1 given by $j^1 f(a) = [f]^1_a$ (notice that $(\pi_1 \circ j^1(f))(a) = \pi_1([f]^1_a) = a$). In coordinates x^i, u, p_i, we have

$$j^1(f)(x^1, \dots, x^n) = (x^1, \dots, x^n, f(x), \frac{\partial f}{\partial x^1}(x), \dots, \frac{\partial f}{\partial x^n}(x)).$$

The space $J^1(\mathbb{R}^n)$ comes equipped with a contact structure given by $\omega = du - p_i dx^i$, or alternatively by the vector fields $D_{x^i} = \partial_{x^i} + p_i \partial_u$ and ∂_{p_i}.

Let $L_f = \{j^1(f)(x) \mid x \in \mathbb{R}^n\} \subset J^1(\mathbb{R}^n)$.

By the computations we did two slides ago, L_f are exactly the integral manifolds that project surjectively to $\mathbb{R}^n$. In this sense, the contact distribution on $J^1(\mathbb{R}^n)$ filters out, from the set of all sections of π_1, the ones that are prolongations (lifts) of functions on $\mathbb{R}^n$.

A first-order PDE is a submanifold in $J^1(\mathbb{R}^n)$

Let us for concreteness consider the PDE $u_{x^1} + uu_{x^2} = 0$. This PDE can be considered as a submanifold

$$\mathcal{E} = \{p_1 + up_2 = 0\} \subset J^1(\mathbb{R}^2).$$

The significance of this submanifold is that if $f(x^1, x^2)$ is a solution to $u_{x^1} + uu_{x^2} = 0$, then we have $L_f \subset \mathcal{E}$ for the submanifold L_f defined on the previous slide.

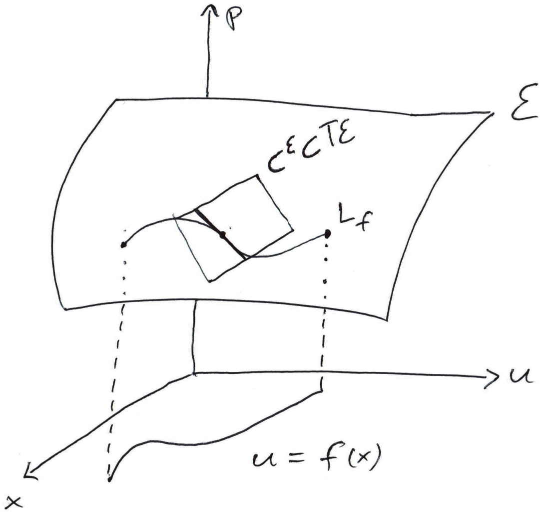
Take for example the solution $u = \frac{x^2}{x^1+1}$ of $u_{x^1} + uu_{x^2} = 0$. Then

$$L_f = \left\{ u = \frac{x^2}{x^1+1}, p_1 = -\frac{x^2}{(x^1+1)^2}, p_2 = \frac{1}{x^1+1} \right\} \subset J^1(\mathbb{R}^2).$$

We see that $(p_1 + up_2)|_{L_f} = -\frac{x^2}{(x^1+1)^2} + \frac{x^2}{x^1+1} \cdot \frac{1}{x^1+1} = 0$.

The contact distribution $\mathcal{C} = \ker(du - p_1 dx^1 - p_2 dx^2)$ on $J^1(\mathbb{R}^2)$ is of rank 4. It restricts to a distribution $\mathcal{C}_q^{\mathcal{E}} = \mathcal{C}_q \cap T_q\mathcal{E}$ on $\mathcal{E}$ (here $q \in \mathcal{E}$). Solutions of $u_{x^1} + uu_{x^2} = 0$ correspond to integral manifolds of the distribution $\mathcal{C}^{\mathcal{E}}$ on $\mathcal{E}$.

A first-order PDE is a submanifold in $J^1(\mathbb{R}^n)$



The Lie Derivative of 1-forms

Let X be a vector field and α be a k-form. We define the interior product $i_X \colon \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$(i_X\alpha)(Y_1, ..., Y_{k-1}) = \alpha(X, Y_1, ..., Y_{k-1}).$$

If β is an l-form, we have

$$i_X(\alpha \wedge \beta) = i_X(\alpha) \wedge \beta + (-1)^l \alpha \wedge i_X(\beta).$$

Using this we can define the Lie derivative of a 1-form with respect to a vector field through Cartan's formula:

$$L_X = di_X + i_X d.$$

The Lie derivative gives the rate of change along integral curves of X. It can actually be defined for arbitrary tensors. In particular, for a function f and vector field Y, the Lie derivative is given by $L_X f = X(f)$, $L_X Y = [X, Y]$.

The Lie Derivative of 1-forms

For example, let $\omega = du - p_1 dx^1 - p_2 dx^2$ and $X = x^1 \partial_u + \partial_{p_1}$. Then

$$i_X \omega = x^1, \qquad d\omega = dx^1 \wedge dp_1 + dx^2 \wedge dp_2$$

and

$$\begin{aligned} & i_X(dx^1 \wedge dp_1 + dx^2 \wedge dp_2) \\ = & i_X(dx^1) \wedge dp_1 - dx^1 \wedge i_X(dp_1) + i_X(dx^2) \wedge dp_2 - dx^2 \wedge i_X(dp_2) \\ = & -dx^1. \end{aligned}$$

We see that

$$L_X \omega = (di_X + i_X d)\omega = dx^1 - dx^1 = 0.$$

We say that X preserves the 1-form ω.

Contact vector fields

A contact vector field on a manifold M with contact structure Π is a vector field X satisfying $[X, Y] \in \Pi$ for every $Y \in \Pi$. If $\Pi = \ker(\omega)$ for a 1-form ω, the condition of X being a contact vector field can be written as

$$L_X \omega = \lambda \omega$$

for some function $\lambda \in C^\infty(M)$.

If $\omega = du - p_i dx^i$, then the general vector field satisfying $L_X \omega = \lambda \omega$ is

$$X_F = F\partial_u - F_{p_i}(\partial_{x^i} + p_i \partial_u) + (F_{x^i} + p_i F_u)\partial_{p_i}.$$

The function F is called the generating function of the contact vector field X. Let's write down some examples for different choices of F:

$$X_{-p_i} = \partial_{x^i}, \qquad X_u = u\partial_u + p_i \partial_{p_i}, \qquad X_{1-x^1 p_2} = x^1 \partial_{x^2} + \partial_u - p_2 \partial_{p_1}.$$

Exercise: Verify that these are contact vector fields by computing $L_X \omega$.

A note on symmetries of PDEs

Consider again the PDE

$$\mathcal{E} = \{p_1 + up_2 = 0\} \subset J^1(\mathbb{R}^2).$$

Definition

An (infinitesimal) symmetry of $\mathcal{E}$ is a contact vector field that is tangent to $\mathcal{E}$, i.e. a vector field X satisfying $X(p_1 + up_2)|_{\mathcal{E}} = 0$.

Two of the contact vector fields from the previous slide are symmetries of $\mathcal{E}$:

$$X_{-p_i} = \partial_{x^i}, \qquad X_{1-x^1 p_2} = x^1 \partial_{x^2} + \partial_u - p_2 \partial_{p_1}.$$

Their flows are transformations on $J^1(\mathbb{R}^2)$:

$$x^i \mapsto x^i + t, \qquad (x^2, u, p_1) \mapsto (x^2 + tx^1, u + t, p_1 - sp_2)$$

We can use these transformations to transform the solution $u = \frac{x^2}{x^1+1}$ to other solutions. For example, using the last of them gives the solution $u = \frac{x^2+tx^1}{x^1+1} - t$.

The thermodynamic identity

The fundamental thermodynamic identity is often written as

$$dU = TdS - PdV$$

where the variables U, T, S, P, V are called inner energy, temperature, entropy, pressure and volume. It can be regarded as a statement of conservation of energy. In that context dU, dS, dV is often talked about as "small changes" in these variables.

By rewriting the identity slightly we recognize a 1-form $\theta = dU - TdS + PdV$. The distribution $\ker(\theta)$ is a contact distribution. The thermodynamic identity can be reformulated by simply saying that a thermodynamical system is a 2-dimensional integral manifold of the distribution θ on $\mathbb{R}^5(U, T, S, P, V)$.

Exercise: Verify that $\theta \wedge d\theta \wedge d\theta = 0$.

Locally, all contact structures look the same

Theorem (Darboux)

Let ω *be a contact form on* M*. Then there exist local coordinates* $x^1, \dots, x^n, u, p_1, \dots, p_n$ *such that* $\omega = du - p_i dx^i$*.*

Exercises

1. Consider the distribution on $\mathbb{R}^4$ spanned by the vector fields $\partial_{x^1}, \partial_{x^2}, e^{x^2}\partial_{x^3}$.
   - Describe it as the kernel of a 1-form $\alpha = a_i dx^i$. I.e. determine the coefficients a^i. Are they unique?
   - Show that the distribution is integrable (using the vector fields, the 1-form, or both).
   - Describe the integral manifolds.
2. Consider the space $\mathbb{R}^4(x, u, p, q)$ with the distribution $\mathcal{C} = \langle \partial_x + p\partial_u + q\partial_p, \partial_q \rangle$.
   - Describe the distribution as the kernel of two 1-forms.
   - Is the distribution integrable?
   - Is this a contact manifold?

   (This space is $J^2(\mathbb{R})$, the space of 2-jets of functions on $\mathbb{R}$. A function $f \in C^\infty(\mathbb{R})$ determines a section of $J^2(\mathbb{R})$ given by $x \mapsto (x, f(x), f'(x), f''(x))$. Its image is a one-dimensional integral manifold of $\mathcal{C}$. A second-order ODE can be thought of as a 3-dimensional submanifold $\mathcal{E} \subset J^2(\mathbb{R})$ and its solutions correspond to integral curves of $\mathcal{C} \cap T\mathcal{E}$.)

Sources

- R.W. Sharpe, *Differential geometry: Cartan's Generalization of Klein's Erlangen Program*, Springer (1997).
- V.I. Arnol'd, *Mathematical Methods of Classical Mechanics*, 2nd ed., Springer-Verlag (1989).
- I.S. Krasil'shchik, A.M. Vinogradov (eds.), *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, AMS (1999)