Thermodynamics and contact geometry
Lecture 1: Vector fields and distributions

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Motivation

V.I. Arnol'd, *Contact Geometry: the Geometrical Method of Gibbs' Thermodynamics*, Proceedings of the Gibbs Symposium: Yale University (1989):

> *Every mathematician knows it is impossible to understand an elementary course in thermodynamics. The reason is that thermodynamics is based—as Gibbs has explicitly proclaimed—on a rather complicated mathematical theory, on the contact geometry. Contact geometry is one of the few 'simple geometries' of the so-called Cartan's list, but it is still mostly unknown to the physicist—unlike the Riemannian geometry and the symplectic or Poisson geometries, whose fundamental role in physics is today generally accepted.*

Motivation

R. Hermann, *Geometry, Physics, and Systems*, M. Dekker (1973):

> *In certain books, thermodynamics is presented as a perfectly "axiomatic," mathematized subject. The reality is quite different; it seems to be a collection of mathematical and physical concepts which are very difficult to make precise, or even to explain in a clear way, at least, in comparison with other areas of physics, such as classical or quantum mechanics. With thermodynamics however, one enters a domain in which the mathematics is easy, at least, on the surface. However, the concepts seem to be very confused, to the mathematician's eye.*

Overview of lectures

Goal

1. Give a differential geometrical overview of thermodynamics.
2. Introduce the relevant concepts from differential geometry.

Plan

- Lecture 1 Vector fields and distributions
- Lecture 2 Differential forms and contact manifolds
- Lecture 3 Thermodynamics
- Lecture 4 Metrics on thermodynamic states
- Lecture 5 Group action and invariants

Smooth manifolds

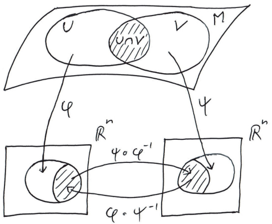
Definition

An n-dimensional manifold M is a (paracompact) Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^n$, i.e. for each $p \in M$ there exists an open set $U \subset M$ containing p and a homeomorphism $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^n$.

We call (U, φ) a chart. If $x^1, \dots, x^n$ are coordinates on $\mathbb{R}^n$, then $x^i \circ \varphi$ are coordinates on M.

In order to define things like derivatives on manifolds, we need additional structure. We say that two charts $(\varphi, U), (\psi, V)$ on M are compatible if $U \cap V = \emptyset$ or if $U \cap V \neq \emptyset$ and the map $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism (it is smooth with a smooth inverse).

Smooth manifolds



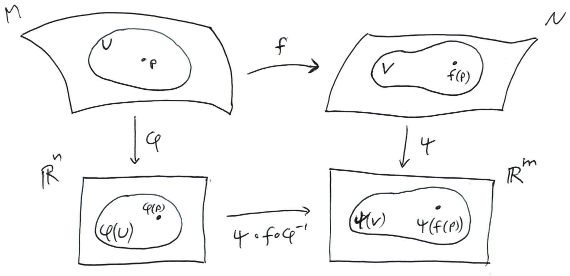
The situation looks like this. On the top we see the manifold M with two open charts U, V, and on the bottom we see the transition map $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$.

Definition

A smooth structure on an n-dimensional manifold M is a maximal collection $\{(U_\alpha, \varphi_\alpha)\}$ of charts that cover M (i.e. $\cup_\alpha U_\alpha = M$) and that are pairwise compatible. A smooth manifold is a manifold equipped with a smooth structure.

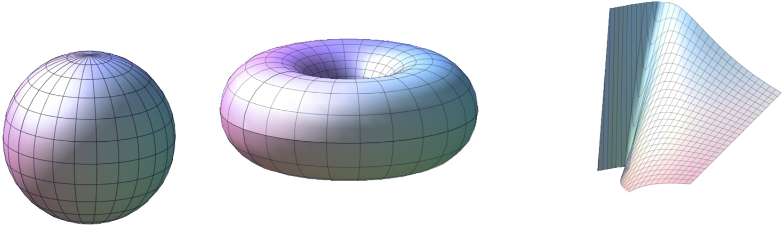
Smooth maps

We say that a continuous map $f\colon M \to N$ between two smooth manifolds is smooth at $p \in M$ if there exist charts $(U \subset M, \varphi)$, $(V \subset N, \psi)$ that are compatible with the smooth structures on M and N, respectively, such that $p \in U$, $f(p) \in V$ and the map $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$. The map f is smooth if it is smooth at every point.



We call f a diffeomorphism if there exists a smooth map $g\colon N \to M$ such that $f \circ g = \mathrm{id}_N$ and $g \circ f = \mathrm{id}_M$. A smooth function on M is a smooth map from M to $\mathbb{R}$.

Examples of smooth manifolds



- $\mathbb{R}^n$, sphere, torus
- The configuration space and phase space of a mechanical system
- 4-dimensional spacetime
- Smooth solutions of differential equations
- Thermodynamic systems in equilibrium

Smooth manifolds

What all of these manifolds have in common, is that they look like $\mathbb{R}^n$ in a local neighborhood. In this sense, all smooth manifolds of the same dimension are locally equivalent (diffeomorphic).

The reason that local differential geometry is still interesting is that one can impose different additional structures on M. These structures are often defined in terms of vectors (elements in the tangent space of M), covectors or, more generally, in terms of tensors.

The focus of this lecture will be distributions. They are smooth subbundles of the tangent bundle of a manifold.

The tangent space

Let M be a smooth manifold and $C^\infty(M)$ be the algebra of smooth functions on it. A derivation on $C^\infty(M)$ at the point p is a linear map $D\colon C^\infty(M) \to \mathbb{R}$ satisfying

- $D(af + bg) = aD(f) + bD(g)$ for all $f, g \in C^\infty(M),\ a, b \in \mathbb{R}$ (linearity),
- $D(fg) = D(f)g(p) + f(p)D(g)$ for all $f, g \in C^\infty(M)$ (Leibniz rule).

The linear combination of two derivations is a derivation: $(aD_1 + bD_2)(f) = aD_1(f) + bD_2(f)$. Thus the space of derivations at $p \in M$ constitutes a vector space. We call it the tangent space of M at p and denote it by T_pM.

A choice of coordinates $x^1, \ldots, x^n$ on M gives a basis on T_pM: $\{\partial_{x^1}|_p, \ldots, \partial_{x^n}|_p\}$. They act on a function $f \in C^\infty(M)$ by $\partial_{x^i}|_p(f) = \frac{\partial f}{\partial x^i}(p)$.

The tangent bundle and vector fields

We define the tangent bundle $TM = \sqcup_{p \in M} T_pM$. Let π denote the projection $TM \to M$. In coordinates, a vector $v \in TM$ can be written as $v = a^i \partial_{x^i}|_p$. Then $\pi(v) = p$.

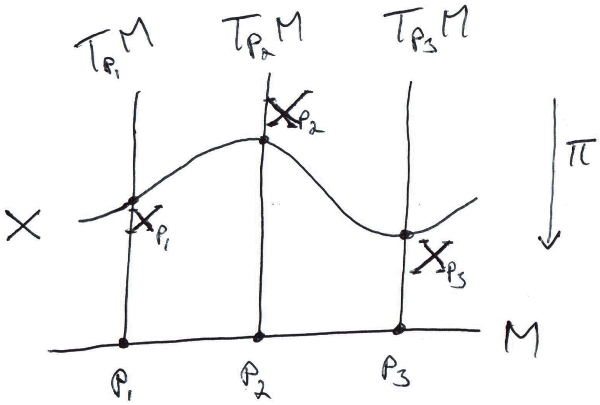
Definition

A vector field is a smooth map $X \colon M \to TM$ satisfying $(\pi \circ X)(p) = p$. We denote the space of vector fields on M by $\mathcal{D}(M)$.

Let ∂_{x^i} denote the vector field defined by $p \mapsto \partial_{x^i}|_p$. The set $\{\partial_{x^1}, \ldots, \partial_{x^n}\}$ defines a frame of TM meaning that at each point p they give a basis of T_pM. Therefore any vector field can be written in coordinates

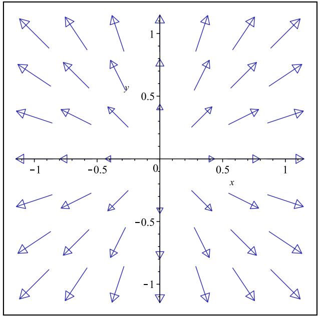
$$X = a^1(x^1, ..., x^n)\partial_{x^1} + \cdots + a^n(x^1, ..., x^n)\partial_{x^n} = a^i(x)\partial_{x^i}.$$

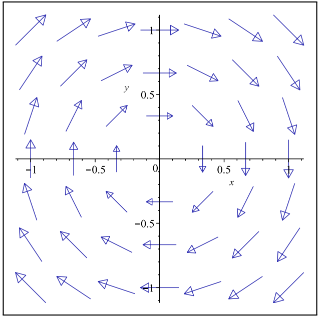
The tangent bundle and vector fields

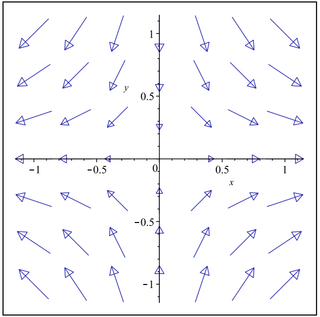


Examples

Some examples of vector fields on $\mathbb{R}^2$ are $x\partial_x + y\partial_y$, $y\partial_x - x\partial_y$, $x\partial_x - y\partial_y$.







Integral curves

We can think of vector fields as being tangent to curves. Given a vector field X on a manifold M, one of the important tasks is to find its integral curves. Let γ be a curve given by a parametrization $x^i(t)$. Then γ is an integral curve to $X = a^i \partial_{x^i}$ if and only if it is a solution to a system of ODEs:

$$\dot{x}^i(t) = a^i(x(t)).$$

This system can always be solved locally.

Examples:

- Consider $X = x\partial_x + y\partial_y$. The equation for the integral curve is $\dot{x}(t) = x(t), \dot{y}(t) = y(t)$ which has solution $(x(t), y(t)) = (x_0 e^t, y_0 e^t)$.
- Consider $X = y\partial_x - x\partial_y$. The integral curves are solutions to $\dot{x}(t) = y(t), \dot{y}(t) = -x(t)$, and they are thus of the form $(x(t), y(t)) = (x_0 \cos(t) + y_0 \sin(t), y_0 \cos(t) - x_0 \sin(t))$.

Exercise: Show that the integral curves can be considered as solutions to $y = Cx$ and $x^2 + y^2 = C^2$, respectively. Describe the integral curves of $x\partial_x - y\partial_y$.

Integral curves

The vector fields can be considered as derivations on functions on our manifold. For example, for $X = y\partial_x - x\partial_y$ we have

$$X(f(x,y)) = yf_x(x,y) - xf_y(x,y).$$

The integral curves of X are given by $x^2 + y^2 - C^2 = 0$. We see that

$$X(x^2 + y^2 - C^2) = 2yx - 2xy = 0.$$

This means that the value of $x^2 + y^2 - C^2$ does not change when moving along the vector field X. In other words, X is tangent to the curve.
Question: What are the integral curves of $\tilde{X} = h(x,y)X$?
Answer: We see that $\tilde{X}(x^2 + y^2 - C^2) = h(x,y)(2yx - 2xy) = 0$, so $\tilde{X}$ and X have the same integral curves as long as $h(x,y) \neq 0$.

If X_p is tangent to the curve γ passing through $p \in \mathbb{R}^2$, then so is $h(p)X_p$.

Distributions

The vector field X from the previous slide gives a vector at each point of $\mathbb{R}^2$. The same is true for $\tilde{X} = h(x,y)X$. These vectors are different, but X_p and $\tilde{X}_p$ lie in the same subspace of $T_p\mathbb{R}^2$.

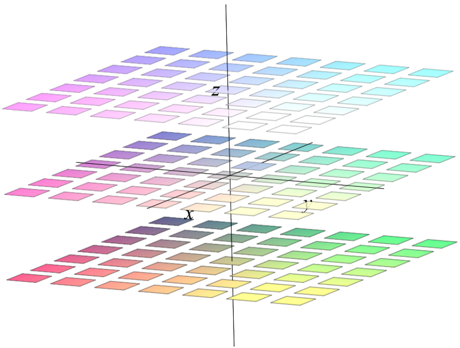
Definition

A distribution on M is a collection of subspaces $\Pi_p \subset T_pM$ that depend smoothly on the point $p \in M$.

By "smooth dependence" we mean that it can be given locally by a set of vector fields: For each point $p \in M$, there exists a local neighborhood $U \subset M$ containing p such that $\Pi_q = \langle X_1|_q, \ldots, X_r|_q \rangle$ for each point $q \in U$. (Here $\langle \cdot \rangle$ denotes the linear span.) In this case we say that $X_1, \ldots, X_r$ span the distribution, and write $\Pi = \langle X_1, \ldots, X_r \rangle$.

If $Y = b^1 X_1 + \cdots + b^r X_r$ for $b_i \in C^\infty(M)$, then $Y_p \in \Pi_p$ and we write $Y \in \langle X_1, \ldots, X_r \rangle$.

An integrable distribution



Integrable distributions

If $\dim \Pi_p = r$ at each point of M, we call Π a regular distribution of rank r. Such a distribution is called integrable if for any point $p \in M$ there exists an open neighborhood U containing p and coordinates on $y^1, \ldots, y^n$ on U such that for each $q \in U$ we have

$$\Pi_q = \langle \partial_{y^1}|_q, \ldots, \partial_{y^r}|_q \rangle.$$

A submanifold $N \subset M$ is called an integral submanifold for Π if $T_pN \subset \Pi_p$ for each $p \in N$.

In particular, if Π is integrable as above, then there exist r-dimensional integral manifolds, and they can be given (locally) by $y^{r+1} = c_1, \ldots, y^n = c_{n-r}$.

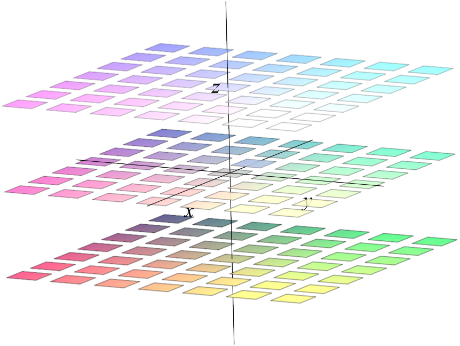
Examples

Example 1: The distribution $\langle y\partial_x - x\partial_y \rangle$ on $\mathbb{R}^2$ is equal to the distribution $\langle h(x,y)(y\partial_x - x\partial_y)\rangle$ for all functions h satisfying $h(x,y) \neq 0$ for every point $(x,y) \in \mathbb{R}^2$. The distribution is regular on the complement of $(0,0) \in \mathbb{R}^2$. The curves $x^2 + y^2 = C^2$ are called integral curves of the distribution. Notice that the distribution is not regular at $x = y = 0$.

Example 2: Consider now $\langle y\partial_x - x\partial_y \rangle$ as a distribution on $\mathbb{R}^3(x,y,z)$. For this distribution, the integral curves are given by $\{x^2 + y^2 = C_1^2, z = C_2\}$.

Example 3: Consider the regular distribution $\langle \partial_x, \partial_y \rangle$ on $\mathbb{R}^3(x,y,z)$. It is of rank 2 and its integral manifolds are given by $z = C$.

A sketch of $\langle \partial_x, \partial_y \rangle$



Another example

Consider the vector fields

$$X = x\partial_y - y\partial_x, \qquad Y = \partial_z$$

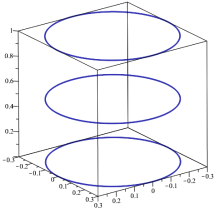
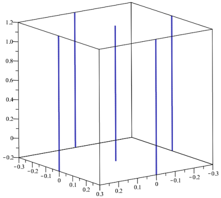
on the manifold $\mathbb{R}^3$ with coordinates x, y, z. Their integral curves are given by

$$\begin{aligned}(x(t), y(t), z(t)) &= (x_0 \cos(t) + y_0 \sin(t), y_0 \cos(t) - x_0 \sin(t), z_0),\\ (x(t), y(t), z(t)) &= (x_0, y_0, z_0 + t)\end{aligned}$$

respectively, or by

$$\{x^2 + y^2 = C_1^2, z = C_2\}, \qquad \{x = C_1, y = C_2\}.$$

Another example

For the vector fields

$$X = y\partial_x - x\partial_y, \qquad Y = \partial_z$$

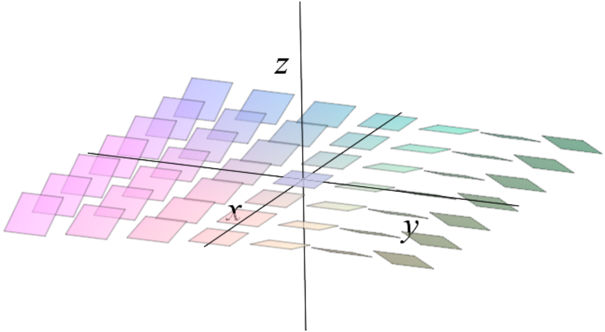
on $\mathbb{R}^3$, consider the distribution $\Pi = \langle X, Y \rangle$. The subspace $\Pi_p \subset T_p\mathbb{R}^3$ is 1-dimensional for $p = (0, 0, z)$ and 2-dimensional at every other point. At each point the vector fields are tangent to the cylinders given by

$$x^2 + y^2 - C^2 = 0,$$

which are the integral manifolds of the distribution.

The contact distribution

The contact distribution on $\mathbb{R}^3$ is given by $\langle \partial_x + y\partial_z, \partial_y \rangle$.



Question: Does it have 2-dimensional integral manifolds? (Imagine what a 2-dimensional manifold around $(0, 0, 0)$ would look like.)

The Lie bracket

The space of vector fields on M can be identified with the space of derivations on the algebra $C^\infty(M)$. For any two vector fields $X, Y \in \mathcal{D}(M)$ and any two functions $f, g \in C^\infty(M)$ we have the vector field $fX + gY \in \mathcal{D}(M)$, so $\mathcal{D}(M)$ is a $C^\infty(M)$-module. We also have another important operation, called the Lie bracket, which is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

In coordinates it looks like this: Let $X = a^i(x)\partial_{x^i}$ and $Y = b^i(x)\partial_{x^i}$. Then

$$\begin{aligned}
[X, Y] &= a^i \partial_{x^i}(b^j \partial_{x^j}) - b^j \partial_{x^j}(a^i \partial_{x^i}) \\
&= a^i b^j_{x^i} \partial_{x^j} + a^i b^j \partial_{x^i} \circ \partial_{x^j} - b^j a^i_{x^j} \partial_{x^i} - b^j a^i \partial_{x^j} \circ \partial_{x^i} \\
&= (a^i b^j_{x^i} - b^i a^j_{x^i}) \partial_{x^j}
\end{aligned}$$

The Lie bracket

The Lie bracket $[\cdot,\cdot]\colon \mathcal{D}(M) \times \mathcal{D}(M) \to \mathcal{D}(M)$ satisfies the following properties:

- $[X, Y] = -[Y, X]$
- $[X, aY + bZ] = a[X, Y] + b[X, Z]$
- $[X, fY] = X(f)Y + f[X, Y]$
- $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$

for all $X, Y, Z \in \mathcal{D}(M)$, all $a, b \in \mathbb{R}$ and every $f \in C^\infty(M)$. **Examples on $\mathbb{R}^3$:**

- $[\partial_x, \partial_y] = 0$
- $[y\partial_x - x\partial_y, \partial_z] = 0$
- $[\partial_x + y\partial_z, \partial_y] = -\partial_z$

Recall that the distributions $\langle \partial_x, \partial_y \rangle$ and $\langle y\partial_x - x\partial_y, \partial_z \rangle$ have 2-dimensional integral manifolds while $\langle \partial_x + y\partial_z, \partial_y \rangle$ does not.

Involutivity

The two distributions $\langle \partial_x, \partial_y \rangle$ and $\langle y\partial_x - x\partial_y, \partial_z \rangle$ may seem special since the the Lie bracket of their generators vanish. However the distribution $\langle \partial_x, \partial_y \rangle$ can equally well be given by

$$\langle a(x,y,z)\partial_x + b(x,y,z)\partial_y, c(x,y,z)\partial_x + d(x,y,z)\partial_y \rangle,$$

for any matrix

$$\begin{pmatrix} a(x,y,z) & b(x,y,z) \\ c(x,y,z) & d(x,y,z) \end{pmatrix}$$

that is nondegenerate at every point. We have

$$[a\partial_x + b\partial_y, c\partial_x + d\partial_y] = (ac_x + bc_y - ca_x - da_y)\partial_x + (ad_x + bd_y - cb_x - db_y)\partial_y.$$

Thus the Lie bracket of the generators is a $C^\infty(M)$-linear combination of the generators. In fact, the special property of $\langle \partial_x, \partial_y \rangle$ is not $[\partial_x, \partial_y] = 0$, but $[\partial_x, \partial_y] \in \langle \partial_x, \partial_y \rangle$.

Involutivity and Frobenius' theorem

Definition

Let $\langle X_1, \ldots, X_r \rangle$ be a regular distribution of rank r. We say that the distribution is involutive if $[X_i, X_j] \in \langle X_1, \ldots, X_r \rangle$ for each $i, j \in \{1, ..., r\}$.

(Consider a distribution $\langle X_1, \ldots, X_r \rangle$ of rank r and assume that $N \subset M$ is an r-dimensional integral manifold. Then the vector fields can be considered as vector fields $Y_1, \ldots, Y_r$ on N. We have $[Y_i, Y_j] \in \langle Y_1, \ldots, Y_r \rangle$.)

Theorem (Frobenius)

A regular distribution Π of rank r on M is involutive if and only if it is integrable.

Recall that Π is integrable if there exists, around each point in M, a coordinate neighborhood (U, y^i) such that

$$\Pi|_U = \langle \partial_{y^1}, \ldots, \partial_{y^r} \rangle.$$

In these coordinates, the distribution has integral manifolds given by $y^{r+1} = c_1, \ldots, y^n = c_{n-r}$ where c_i are arbitrary constants.

Revisiting our examples

We have been looking at three examples of distributions: $\langle \partial_x, \partial_y \rangle$, $\langle \partial_x + y\partial_z, \partial_y \rangle$, $\langle y\partial_x - x\partial_y, \partial_z \rangle$. We noticed that

- $[\partial_x, \partial_y] = 0$,
- $[y\partial_x - x\partial_y, \partial_z] = 0$,
- $[\partial_x + y\partial_z, \partial_y] = -\partial_z$.

By Frobenius' theorem, the first two are integrable while the last one is not (as we could guess from the picture). The first one is already in "Frobenius coordinates". For the second distribution, if we use cylindrical coordinates

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z,$$

then the distribution can be given by $\langle \partial_\theta, \partial_z \rangle$. Note that the cylindrical coordinates are only good away from the z-axis.

One-dimensional integral manifolds of the contact distribution

The contact distribution $\langle \partial_x + y\partial_z, \partial_y \rangle$ is not involutive and does therefore not admit 2-dimensional integral manifolds. But it does admit 1-dimensional integral manifolds. Let us parametrize a curve by $z = f(x), y = g(x)$. Its tangent vectors are spanned by $\partial_x + f'(x)\partial_z + g'(x)\partial_y$.

Question: What does it take for $\partial_x + f'(x)\partial_z + g'(x)\partial_y$ to be in the distribution?
Answer: The condition $\partial_x + f'(x)\partial_z + g'(x)\partial_y = a(\partial_x + g(x)\partial_z) + b\partial_y$ implies that $a = 1, b(x) = g'(x), g(x) = f'(x)$ with no conditions of f.

Integral curves are thus given by $z = f(x), y = f'(x)$ (if they can be parametrized by x). We also have some special integral curves: $x = C_1, z = C_2$.

$(2,3,5)$-distributions in $\mathbb{R}^5$

Consider the distribution[1]

$$\langle X = \partial_q, Y = \partial_x + p\partial_y + q\partial_p + f(q)\partial_z \rangle.$$

This is another example of a noninvolutive distribution. We have

$$[X,Y] = \partial_p + f'(q)\partial_z = Z,$$
$$[X,Z] = f''(q)\partial_z, \qquad [Y,Z] = -\partial_y.$$

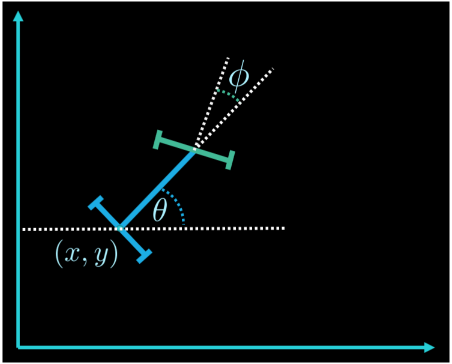
By taking Lie brackets, we get a sequence of distributions

$$\langle X, Y \rangle, \qquad \langle X, Y, [X,Y] \rangle, \qquad \langle X, Y, [X,Y], [X,[X,Y]], [X,[Y,Z]] \rangle.$$

Exercise: Check that my calculations are correct. For suitable choices of f these distributions will be regular of rank 2, 3 and 5, respectively. What conditions must f satisfy for this to be true?

[1] See for example [An, Nurowski, arxiv.org/1302.1910, (2016)].

Parking a car[2]



The configuration space of a car of length L is $\mathbb{R}^2 \times S^1 \times (-\pi/2, \pi/2)$ with coordinates x, y, θ, ϕ, but the driver can move in this space only by steering or driving. He is thus restricted to move tangent to the planes spanned by

$$S = \partial_\phi, \qquad D = \cos(\theta)\partial_x + \sin(\theta)\partial_y + \frac{\tan(\phi)}{L}\partial_\theta.$$

[2] I found this example at mathoverflow.net/questions/66578.

Parking a car

We compute the Lie bracket:

$$[S, D] = \frac{1}{L\cos^2(\phi)}\partial_\theta$$

Thus $\langle S, D\rangle$ is not integrable. Notice that the vector field $[S, D]$ corresponds to a rotation. The Lie bracket of this with D is

$$[[S, D], D] = \frac{1}{L\cos^2(\theta)}(\cos(\phi)\partial_y - \sin(\theta)\partial_x).$$

The vector fields $S, D, [S, D], [[S, D], D]$ span the tangent space of the configuration space. This explains (by the Chow-Raschevsky theorem) how the driver is able to bring the car to any configuration, even with the constraints that he has.

The last example in this lecture

Consider the distribution $\langle X = x\partial_y - y\partial_x, Y = x\partial_z - z\partial_x \rangle$. Let us check where it has rank 2, i.e. where the matrix

$$\begin{pmatrix} -y & x & 0 \\ -z & 0 & x \end{pmatrix}$$

is of rank 2. From looking at the last 2×2-minor, we see that the condition is $x \neq 0$.

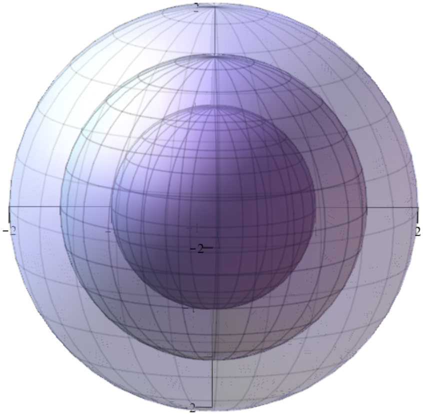
Question: Is this distribution involutive? If it is, what are the integral manifolds?

$$[X, Y] = -y\partial_z + z\partial_y = \frac{z}{x}X - \frac{y}{x}Y$$

If we stay away from the plane given by $x = 0$, the distribution is involutive.

The last example in this lecture

Integral manifolds are then half spheres. If we consider the rank-2 distribution $\langle X, Y, [X,Y] \rangle$, the integral manifolds are spheres.



Sources

Textbooks on differential geometry

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