A bridge in between Poisson Geometry and Noncommutative Geometry through quantization processes.

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Outline

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- Poisson Geometry
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Motivations

The mathematics of quantization are, widely, inspired in physics, of course. The concept of "quantization" and "quantum systems" was born in physics at the beginning of the 20th century. In those times, physics suffered a strong revolution that reached mathematics in different ways. In particular, the mathematics of quantization is an attempt to formalize the process of "canonical quantization" used by physicist, in, pretty much, the same fashion semi Riemannian Geometry is used to rigorously study General Relativity. Now these ideas have its own sense in mathematics and have gave place to new perspectives and techniques to study Geometry.

Quantization in physics

One of the main differences we find in between Quantization in Physics and Quantization in Mathematics is exactly the same we have in between Physics and Mathematics themselves: **Physics has ontological meaning, Mathematics is purely logical**. In Physics, a quantum system is a system of subatomic particles, and "quantum physics" refers to the series of laws and principles that govern the behaviour of subatomic particles. In the first try, physicist applied all they know of classical mechanics to calculate trajectories and positions of the subatomic particles but enormously failed. In short words, a completely new model, proposed by Max Planck, to face these problems triumphed in every calculation and is considered the first treatise ever in Quantum Physics. The key idea was to handle energy not as a continuous but as little "package" or "quanta", as he called it. This idea of treat energy as quantum packets (more like discrete packets) solved the problem and open a completely new branch of physics.

In addition, one of the main features of these new "quantum systems" is that now you have to treat all classical observables as operators. And this is the key idea to quantize in the canonical sense, just promote every classical observable to an operator and include the Schrodinger Equation (like you introduce Newton's Equation of force in classical systems). More in detail, these operators act on a Hilbert space, so, what we have done is take the algebra of classical observables of a classical system to an algebra of operators in a Hilbert space, resulting in a quantum system.

Poisson Geometry

To begin, let's consider an associative algebra with unit $\mathcal{A}$ and introduce the bilinear form

$$\{,\}: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

which satisfies, $\forall\ f, g, h \in \mathcal{A}$,

- i) $\{f, g\} = -\{g, f\}$ (Antisymmetry)
- ii) $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$ (Jacobi's identity)
- iii) $\{f, gh\} = g\{f, h\} + h\{f, g\}$ (Leibniz rule)

The bilinear form $\{,\}$ is called **Poisson Bracket**. Worthy remarkable, due to properties *i*) y *ii*), $\{,\}$ it is, also, a **Lie Bracket**, then we can say that *a Poisson Bracket is a Lie Bracket that satisfies Leibniz rule.*

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What we are doing here is to introduce $\{,\}$ over the algebra of smooth functions of the manifold. To this point we have done nothing but introduce language. What is the main interest to give such definitions at all? The quickest answer is: **Hamilton Equations**.

Property iii), also known as *derivation rule*, implies that $\{f, -\}$ induce a vector field. Thus, we define

Definition.(Hamiltonian vector field)

$X_H := \{H, -\}$
X_H is called **Hamiltonian vector field** of the **Hamiltonian** H.

The canonical example over $\mathbb{R}^{2n}$ with coordinates $\{q_1, \ldots, q_n, p_1, \ldots, p_n\}$ is to define the Poisson Bracket by $\{f, g\} \equiv \sum_{i=0}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$

Then, $X_H = \sum_{i=0}^{n} \left(\frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right)$ and Hamilton Equations are written as

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

This is, the differential equations defined through the Hamiltonian vector field are the Hamilton equations, then $(q(t), p(t))$ is an integral curve of X_H iff the equations are satisfied.

In addition, we have, $\forall f \in C^\infty$

$$\frac{df}{dt} = \{f, H\}$$

So we can express Hamilton equations as

$$\dot{q} = \{q, H\}, \quad \dot{p} = \{p, H\}$$

Another important aspect of the Poisson structure becomes evident when we notice is possible to define **non degenerate** Poisson Brackets, this is

$$\forall g \in C^{\infty}(\mathcal{M}) \;\; \{f, g\} = 0 \Leftrightarrow f = 0$$

So we can define Poisson Brackets through **symplectic forms** as

$$\{f, g\} = \omega(X_f, X_g)$$

Then, any **symplectic manifold**[1] is Poisson.

[1] Remember a symplectic manifold is a manifold $\mathcal{M}$ equipped with a non degenerate, antysymmetric, 2-form ω we call **symplectic form**.

Noncommutative Geometry

To begin this section let me introduce a very important theorem used in Algebraic Topology. Start by considering a topological space X.
Is not hard to prove that the set of analytic functions $C^{\infty} = \{f \mid f\colon X \to \mathbb{C}\}$ satisfy:

i) Forms a Banach space with $\| f \|_{\infty} = sup(\{| f(x) \| x \in X\})$ (supremum norm)

ii) Forms a commutative algebra with unity with the pointwise product of functions.

$$(fg)(x) = f(x)g(x)$$

iii) They posses an *involution*, for this case, given by the usual conjugation of complex numbers.

$$\begin{aligned} *\colon C^{\infty}(X) &\to C^{\infty} \\ f &\mapsto *(f) \equiv f^*(x) := \overline{f(x)} \end{aligned}$$

This is,they form an involutive Banach algebra[2], also known as C^*-**algebra**.

[2] Not forgetting that $\| f \|_{\infty}^2 = \| f^* f \|_{\infty}$

The main idea is based on take the algebra of functions over the topological space and notice that it has a C^*-algebra structure. In addition, if we change the perspective and we take some C^*-algebra, the set of **characters**

$$\chi \colon G \to \mathbb{K}$$

can be promoted to a topological space.
We should remember that any algebra (including C^*-algebras) has a underlying vector space. Then, a **representation** of some group G in V is a function $T \colon G \to Aut(V)$[3]. Thus, a character is the function that assigns elements of the group with elements of the field through $\chi(g) = Tr(T(g))$ (the trace of the representation of g). This set of characters does admits non trivial topologies and, because of that, it is said that we it can be promoted to a topological space.

[3] Usually, it is defined over GL(V), the set of invertible endomorfisms, this is, the automorphisms.

In our example, the field is $\mathbb{C}$; then, it is possible to prove that the set of characters associated to the C^*-algebra induced by $\mathbb{C}$, promoted to a topological space (with the weak topology-*) is identified with the original space X, specifically, with its algebra of analytic functions.

[4] The opposite category is the same category but with the morphisms inverted, $f: A \to B$ now is $f: B \to A$.

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This correspondence in between topological spaces and C^*-algebras is the idea behind the Gelfand duality theorem, which writes

Theorem. (Gelfand duality)

The pair of functors

$$C^*Alg^{opp}_{comm} \leftrightarrows Top_{cpt}$$

is an **equivalence of categories**.

Here, $C^*Alg^{opp}_{comm}$ stands for the **opposite category**[4] to C^*Alg_{comm}, which is the category of commutative C^*-algebras. And Top_{cpt} is the category of compact topological spaces.

[4]The opposite category is the same category but with the morphisms inverted, $f: A \to B$ now is $f: B \to A$.

This theorem allow us to make an identification in between topological properties and algebraic properties, as is shown in the table here below

Topology	C^*-**algebra**
Locally compact topological space	Commutative C^*-algebra
Continuous functions	Algebra homomorphisms
Compactness	Existence of an identity
Disjoint union	Direct sum
Cartesian product	Tensor product

With this formalism on the ground, finally, we can talk about the brilliant idea that **Alain Connes** had of research what kind of topological spaces could be related to **non commutative** C^*-algebras, for instance, something we can call **non commutative topological space**. As well, has become common call them **quantum groups**. This is the main idea behind **Noncommutative Geometry**, to study this topological spaces associated to non commutative algebras of functions.

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A simple way to think about it is remember when we were in elementary and we learn Analytic Geometry. In those times we were capable of deduce the geometric properties of a curve and how does it look in the plane just by taking a look to the associated algebraic equation. Same happened when we look a curve in the plane and we could know the roots of the associated equation, among others algebraic aspects; this is, from algebraic information we deduced geometric information, and vice versa.

Quantization

Plausibly, those initiated in **Quantum Mechanics** have grasp the idea of the bridge that exist in between Poisson Geometry and Noncommutative Geometry. The reason why noncommutative spaces are often labelled as "quantum" is because it is possible to construct them by means of **quantization**.
Symplectic Geometry, in general, Poisson Geometry is widely used to study classical physical systems[5]. This study is based in the algebra of commutative functions over the manifold, $C^\infty(\mathcal{M})$. Nevertheless, in Quantum Mechanics Hilbert spaces and self adjoint operators are used, whose algebra is, essentially, non commutative.

[5] In fact, a big part of the development of this theory is due to the work in classical mechanics.

Dirac noticed, time after his first formulation of Quantum Mechanics, that the algebraic description given by the Poisson Brackets over a manifold $\mathcal{M}$ is crucial to understand the process of quantization. In particular, he contrasted the algebra of operators, which is non commutative, over a Hilbert Space $\mathcal{H}$ with the algebra of functions over the manifold in such a way that quantization must be understood as a morphism that takes functions on the manifold and gives operators in a Hilbert space, this is, a relation in between algebraic structures.This way, we can understand the algebra of self adjoint operators in a Hilbert space as the non commutative maximal space associated to the algebra of functions in the symplectic manifold. By doing this, Dirac gave a series of conditions that must be taken as starting point to any quantization theory.

$$Q\colon C^{\infty}(\mathcal{M}) \to End(\mathcal{H})$$

- i) $\mathbb{R}$-lineality $Q(rf+g) = rQ(f) + Q(g)$
- ii) Normalization $Q(1) = 1_{\mathcal{H}}$
- iii) Hermiticity $Q(f)^* = Q(f)$
- iv) Dirac quantization condition: $[Q(f), Q(g)] = -i\hbar Q(\{f,g\})$
- v) Irreducibility: If $\{f_k\}_{k=1}^{n}$ is a complete set of functions, then $\{Q(f_k)\}_{k=1}^{n}$ is a complete set of operators.

In **Deformation Quantization**, the algebra $C^\infty(\mathcal{M})$ is replaced by the algebra $\mathcal{A}_\hbar \equiv C^\infty(\mathcal{M})[\hbar]$ of power series in $\hbar$ of elements of $\mathcal{A}_\hbar$. Then, $\mathsf{f} \in \mathcal{A}_\hbar$ has the form

$$\mathsf{f} = \sum_{k=0}^{\infty} \hbar^k f_k$$

This idea is motivated in analogy to the concept of **symbol**[6] in the theory of **pseudo differential operators**. A **deformation** is the pair $(\mathcal{A}_\hbar, \star)$ where

$$\star \colon \mathcal{A}_\hbar \otimes \mathcal{A}_\hbar \to \mathcal{A}_\hbar$$

Two very specific examples of this process are the *Clifford Algebra* which can be seen as the deformation of the *Exterior Algebra*, and the *Weyl Algebra* which can be seen as the deformation of the *Symmetric Algebra*

[6] A symbol is a polynomial that represents a differential operator, giving place to pseudo differential operators.

Final Remarks

The the starting point of any classical theory of physics is the triplet $(\mathcal{A}, \{,\}, H)$, with $\mathcal{A} = C^{\infty}(\mathcal{M})$. The starting point in Noncommutative Geometry is the **spectral triple** $(\mathcal{C}^*, \mathcal{H}, D)$, where $\mathcal{C}^*$ is an involutive algebra and $\mathcal{H}$ is a Hilbert space, together with the self adjoint operator D. Both, linked via quantization map. This is, **the deformation of the algebra of smooth functions in a Poisson manifold corresponds with a noncommutative space, also called, quantum group**. From this now we know why some authors talk about *QuantumGeometry*, just a different name for Noncommutative Geometry.

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