Lattice Point Geometry: Pick's Formula and Minkowski's Theorem

Dmitry N. Oskorbin

Altai State University

Main Theorems

Pick's Formula

Let P be a polygon in the plane with its vertices at lattice point. Then the area $A(P)$ is given by

$$A(P) = \frac{1}{2}B(P) + I(P) - 1,$$

where $B(P)$ is the number of lattice points on the boundary of P, $I(P)$ is the number of lattice points in the interior of P.

Minkowski's Theorem

Let R be bounded, convex region in $\mathbb{R}^2$ that is symmetric about the origin and having area greater then 4. Then R contains an integer point other than the origin.

Lattice

A set L of of points in $\mathbb{R}^n$ is called a lattice if
1. L is a group under vector addition.
2. Each point in L is the center of a ball that contains no other points of L.

- Triangulation
- Finding the area of primitive triangle
- Application of Euler's formula

Triangulation

Lemma 1

Every n – gon P $(n \geq 4)$ has a interior diagonal.

Lemma 2

Every n – gon P $(n \geq 4)$ can be dissected into $k - 2$ triangles, each of which has vertices that are vertices of P, by means of nonintersecting diagonals.

Corollary

Every lattice polygon can be dissected into primitive lattice triangles.

Visible points

Definition

A lattice line is a line in $\mathbb{R}^2$ that passes through at least two lattice points. Let l be a lattice line through the origin. Then the visible points of l are the two non-zero lattice points on l with minimum positive distance to the origin.

Theorem

A lattice point $p = (m, n)$ is visible iff m, n are relatively prime.

Primitive parallelograms

Proposition 1

A primitive parallelogram has area 1.

Proposition 2

A lattice parallelogram P spanned by linearly independent vectors $v, w \in Z^2$ is primitive iff v, w is a basis for Z^2.

Corollary

A primitive triangle T has area $A(T) = \frac{1}{2}$.

End of the proof

Pick's Formula

Let P be a polygon in the plane with its vertices at lattice point. Then the area $A(P)$ is given by

$$A(P) = \frac{1}{2}B(P) + I(P) - 1,$$

where $B(P)$ is the number of lattice points on the boundary of P, $I(P)$ is the number of lattice points in the interior of P.

Lemma (Euler)

If G is a connected, planar graph with v vertices, e edges, and f faces, then $v - e + f = 2$.

Convex regions in $\mathbb{R}^2$

Theorem

Let R be a bounded, convex region in $\mathbb{R}^2$. Then

$$L(R) \leq A(R) + \frac{1}{2}p(R) + 1,$$

where $L(R)$ is the number of lattice points into R, $p(R)$ is the perimeter of R

Minkowski's Theorem

Let R be bounded, convex region in $\mathbb{R}^2$ that is symmetric about the origin and having area greater then 4. Then R contains an integer point other than the origin.

Lemma (Blichfeldt)

If R is a bounded set in $\mathbb{R}^2$ with area greater then 1, then R contains two distinct points (x_1, y_1), (x_2, y_2) such that the point $(x_2 - x_1, y_2 - y_1)$ is an integer point in $\mathbb{R}^2$.

Application of Minkowski's Theorem

Lemma

Let p be a prime of the form $4k + 1$. Then there exists an integer c for which $c^2 + 1 \equiv 0 (mod p)$.

Theorem

Let p be a prime of the form $4k + 1$. Then p can be written as the sum of the squares of two positive integers.