Neurogeometry of vision . Lecture IV . Primary visual corterx

Dmitri Alekseevsky

Institute for Information Transmission Problems, Moscow,Russia and Faculty of Science University of Hradec Kralove, Rokitanskeho 62, Hradec Kralove, 50003, Czech Republic Spring School in Topology and Geometry, Gradec Karlove, May 2018

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Lecture IV Primary visual cortex

1.Retinotopic conformal map from retina to LGN and V1 cortex 2.Architecture of V1 cortex (minicolumnes and columns, simple and complex cells, field of directions. pinwheels)

- 3. Shift of receptive fields and etc Principly by E. Gombrich.
- 4. Contact Petitot model of V1 cortex and Legendrian lift.
- 5. Hypercolumns. Hubel-Wiesel definition of a hypercolumn.
- 6. Bressloff-Cowan spherical model of hypercolumn.
- 7. Symplectic Petitot -Citti-Sarti model of V1 cortex and parametrisation of set of simple cells by the similarity group $Sim(\mathbb{R}^2)$.
- 8. Lift of input function to similarity group and R. Duits theory of score. Generalisation.
- 9. Unification of Bressloff-Cowan and Petitot-Citti-Sarti models.

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Retinotopic (topographic)map from retina to LGN (lateral ginicular nuclea) and visual cortex VI

(Schwarz) The map is conformal and is given by a function of the form

$$z\mapsto F(z)=\lograc{z+a}{z+b}.$$

The module |F(z)| is the magnification.

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1.Primary visual cortex V1 is a surface of depth 1.5-2 mm which consists of 6 layers. Each layer consists of columns of cells which has approximately the same receptive field.

2.Hubel and Wiesel proposed a classification of VI cells into simple and complex cells.

3. Simple cell acts as Gabor filter (anisotropic Gauss filter). All simple cell from a (regular) column at a point $z \in V1$ acts as approximately identical Gabor filters with center at the corresponding point $z \in R$ and detect contour through point z with fixed "orientation" (i.e. direction on retina R, measured by the angle $\theta \in [0.\pi)$ (up to 15-20 %)).

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4. The field of orientations (=directions) defines a 1-dimensional distribution $\Gamma = \ker \eta$, $\eta \in \Omega^1(S^2)$. The 1-form η has many isolated singularities (called pinwheels) - points where $\eta = 0$. Corresponding singular column contains simple cells of any orientation and hence a singular column detects a contour with any orientation. So singular columns acts as a watch towers.

5. One of the aim of eyes movement is to produce the shift of the image on retina such that contours intersect pinwheels and will be detected by more cells.

Pinwheels



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Field of directions with hexagonal symmetry, which is a solution of generalized Swift-Hohenberg equation (F. Wolf et al.)

Pinwheels and hexagonal lattice





Pinwheels and horizontal connections



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Shift of RF and remapping (Jean-Rene Duhamel; Carol L. Colby; Michael E. Goldberg,1992)

In a seminal paper, Duhamel et al. described the shift or RF of many neurons in macaque lateral intraparietal area (LIP). Assume that the RF of a neuron before saccade covers the retina image \bar{A} of a point A and after a saccade the retina image \bar{A}' of another point A'. Then 100 ms before a saccade, the neuron detects stimuli at the locations \bar{A}' . This process constitutes a remapping of the stimulus from the retina coordinates with the initial fixation point to those of the future fixation point. The process is governed by a copy of the motor command (corollary discharge)

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Remapping of retina images



Fig. 1. Remapping of the visual representation in parietal cortex. Each panel represents the visual image at a point in time relative to a sequence of oculomotor events. Receptive field of a parietal neuron, dashed circle; center of current gaze location, solid circle; and coordinates of the cortical representation, cross hairs.

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Fixation fields after first 6 saccades



Fixation fields after first 6 saccades

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The global pattern in environments such as a forest, beach or street scene enables us to predict more-or-less what we will see, based on the order and redundancy in the scene and on previous experience with that type of environment.

(Recall a presentation of a walking person in low cost cartoon).

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Realisation of the Gombrich principle. Remapping as a conformal transformation.

We will use the Moebius projective model of the eye sphere as the projectivisation $S^2 = PV_0 \subset PV$ of the isotropic cone V_0 of the Minkowski space V. The central projection

$$\pi_F: PV \to S^2, A \mapsto \ell(A, F) \cap S^2$$

is defined as the point of intersection of the line $\ell(A, F)$ with sphere , different from F.

The Lorentz group SO(V) is the group of projective transformations of PV which preserves the sphere $S^2 = PV_0$

Note that in coordinate system, fixed w.r.t. the eye sphere, a saccade corresponds to an affine transformation $\rho \in SO(V)$ of PV (rotation) which preserves S^2 (but does not preserve F). It maps a plane Π , defined by three points $A_i \in PV$, i = 1, 2, 3 to a plane Π' defined by by points $A'_i = \varphi(A_i)$. Let $\pi_F : PV \to S^2$ the cental projection and $(A_i) = \pi_F(A_i), (A_i)' = \pi_F(A'_i)$ are images of these points on retina before and after a saccade.

There exists unique Lorents transformation φ of the projective space PV which preserves the eye sphere and the point $F \in S^2$ such that

 $\varphi(\Pi) = \Pi', \ \varphi(A_i) = A'_i.$

It is determined by its restriction to S^2 which is the unique conformal transformation with transforms the images $\bar{A}_i := \pi_F(A_i)$ of points before and after saccade and describe the remapping of the retina image after saccade.

In 1989 , W. Hoffman stated that the primary visual cortex is a contact bundle .

The realization of this idea was done by J. Petitot.

Under approximation that all points are centers of pinwheel,

J.Petitot concludes that points of VI cortex are parametrized by the "orientation" i.e. directions in the retina sphere S^2 , i.e. points of the projectivised (co)tangent bundle $PT^*S^2 = PTS^2$. It is the contact bundle of S. Lie with the natural contact structure $D = \pi_* \ker \theta \subset PT^*S^2$, which is defined by the projection $\pi : T^*S^2 \to T^*S^2$ of the canonical distribution $\ker \theta \subset T(T^*S^2)$. Here $\theta = p_i dq^i$, $\theta_{\xi}(X) = \xi(\pi_*)(X)$ is the tautological Liuville 1-form on T^*S^2 . Simple cells in V1 cortex detect the orientation (= directions) of a contour. So they determine the (Legendrian) lift of the contour to the horizontal curve $\bar{C}_T \subset PT^*(S^2)$.

So, according to Petotot model, VI cortex is the contact bundle PTS^2 with the canonical contact structure and simple cells determine the Legendrian lift of contours in S^2 to PT^*S^2 . If (x, y) are coordinates in retina S^2 such that contours are described as y = y(x), then the contact manifold PTS^2 can be locally identified with the manifold $J^1(\mathbb{R})$ of 1-jets of functions with coordinates $(x, y, p = \frac{dy}{dx})$ and the contact form $\Theta = dy - pdx$. The contact manifold $J^1(\mathbb{R})$ is identified with the Heisenberg group $Heis_3$ or with the group $E(2) = SO_2 \cdot \mathbb{R}^2$ of Euclidean motions of the plane with left invariant contact structure.

A natural generalization of the Petitot's model was proposed by G.Citti, A.Sarti, and J.Petitot. They assume that a simple cell is characterized not only by the point $z \in R \subset S^2$ (the center of the receptive field) and the orientation θ , but also by scaling σ - the intensity of the reply on the same stimulus.

Then the space of simple cells in VI cortex (= Gabor filters) are parametrized by the similarity group $B^- = Sim(E^2) = \mathbb{R}^+ \cdot SO_2 \cdot \mathbb{R}^2_- = G_0 \cdot G_-$ where $z = (x, y) \in G_$ indicates the center of a Gabor filter, $\theta \in SO_2$ its orientation and $\sigma \in \mathbb{R}^+$ is the scale - the intensity of reply. Locally this space can be identified with the cotangent bundle T^*S^2 or the principal $G_0 = CO_2 = \mathbb{R}^+ \cdot SO_2$ -bundle of conformal frames over S^2 with the standard symplectic structure.

Huble and Wiesel proposed a deep and very productive notion of hypercolumns in V1 cortex. Given a system of local parameters (for example, orientation and ocular dominance or orientation and spatial frequency). A hypercolumn (or, module) is defined as a minimal collection of (regular) columns, containing simple cells which measure any possible value of these parameters and which is sufficient to detect the local structure of an image.

Applying this notion to orientation and ocular dominance , they proposed a famous ice cube model of V1 cortex. Now this notion is applied also for V2 cortex.

Usually, the area of hypercolumns is $1 - 2mm^2$.

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Bressloff and Cowan considered a parametrisation of simple cells by two parameters:

orientation θ and spatial frequency $p \in [p_L, p_H]$, and proposed a model of hypercolumn as a sphere associated with two pinwheels S, N, which correspond to local minimum and local maximum of the spatial frequency p.

Spherical model of hypercolumn by Paul Bressloff Рч Jack Cowan

Hypercolumns as spheres.

More precisely, they introduce the normalized logarithm $\sigma = \pi \frac{\log(p/p_L)}{\log(p_L/p_L)} - \pi/2$ of the space frequency as a new parameter and propose to consider this parameter σ together with the orientation θ as spherical coordinates - latitude and longitude on a sphere, which represents a hypercolumn. Two pinwheels corresponds to north N and south S poles of the sphere where the latitude σ take value $\pi/2$ and $-\pi/2$ respectively and the spatial frequency take maximal and minimal values. In these points, the longitude = orientation is not defined. It is consistent with the fact, that the orientation is not defined in pinwheeels, since there are simple cells which measure contour of any orientation.

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Evolution of an excitation in a hypercolumn is described by the Wilson-Cowan's equation

$$\partial_t u(\theta,\sigma,t) = -u(\theta,\sigma,t) + \int_0^{\pi} \int_0^{\pi} W(\theta,\sigma|\theta',\sigma') \rho(u(\theta',\sigma',t)) d\nu + h(\theta,\sigma))$$

where W is the density of interaction between two hypercolumns, ρ is a sigmoidal function , h is a stimulus from LGN. The authors assume that the weight function $W \in C^{\infty}(S^2 \times S^2)$ is SO_3 -invariant. Then it can be described in terms of spherical harmonics (a sum $\sum_{n=0}^{\infty} W_n \sum_{-n}^n (Y_n^m)^*(\theta, \varphi)(Y_n^m)(\theta', \varphi')$ of products of spherical harmonics). The simplest example is a function of the Riemannian distance. There is no argument why the sphere-hypercolumn must have a standard SO_3 -invariant Riemannian metric. So at the end of the paper, the authors consider a generalization of the model to the case of non invariant metric.

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We propose a modification of the Bressloff-Cowan model of hypercolumn and consider a model of hypercolumn with spherical coordinates θ (orientation) and φ (defined by Bressloff-Cowan function of spatial frequency) as a conformal sphere. Consider the Riemann model of conformal sphere $S^2 = \hat{\mathbb{C}}$ with two distinguished points S = 0 and $N = \infty$ and complex coordinate $z = x + iy \in S^2 \setminus \{\infty\}$ and $w = \frac{1}{z} \in S^2 \setminus \{0\}$ and the action of the conformal group $G = SL_2(\mathbb{C}) = G^- \cdot G^0 \cdot G^+$. The stability subgroups are $G_S = B_+ = G^0 \cdot G^+$, $G_N = B^- = G^- \cdot G^0$. The points $S = \{0\}, N = \{\infty\}$ correspond to pinwheels, which give minimum and maximum values of the spatial frequency p.

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The stability subgroup $G_N = B^- = G^- \cdot G^0 \simeq Sim(E^2)$ acts transitively on $S^2 \setminus N$ and acts on the tangent space $T_S S^2$ of the south pole via stereographic projection as the group of homotheties $z \mapsto az + b$ with generators $\partial_x, \partial_y, x\partial y - y\partial_x$ (rotation) and $z\partial_z = x\partial_x + y\partial_y$ (dilatation). According to Petitot-Citti-Sarti model, we may parametrize (locally) simple cells in the neighborhood of the point S by this group (i.e. parallel translations, rotations and dilatations). Such neurons detect images with low level of spatial frequency.

Similarly, neurons in a neighborhood of the north pole N are parametrized (locally) by the points of the stability subgroup $G_S = B^+ = G^0 \cdot G^+ \simeq Sim(E^2)$. They detects images with high level of spatial frequency. Note that the orbit of 1-parameter subgroup \mathbb{R}^+ generated by dilatation $z\partial_z = x\partial_x + y\partial_y$ of the sphere are meridians (that is coordinate lines of φ or spatial frequency p) and orbits of subgroup S^1 of rotation corresponds to parallels of the sphere (coordinate lines of orientation θ .)

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Stereographic projection



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 $T_sS^2 = \{z\}$ $T_NS^2 = \{w=1/z\}$

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We will state the following general principle of invariancy: Let G be a group of transformations of a space V and $\mathcal{O} = Gx$ an orbit. If observers are distributed along the orbit \mathcal{O} , the information, about some process, which they detect and send to some centre is invariant w.r.t. the group G. Probably, the simple cells of a hypercolumn near S send the information to singular pinwheel column S (center of low spatial frequency images) and cells near N send information to pinwheel N (the center of high frequency).

The system of simple cells of a hypercolumn S^2 is invariant w.r.t. subgroup $G^0 = G_S \cap G_N = B^- \cap B^- = CO_2 = S^1 \times \mathbb{R}^+$. Assume that a local retina image remains inside hypercolumn during fixation eye movements. Then the information, which is detected by all simple cells of a hypercolumn and send to the next level, is invariant with respect to the group G^0 . Moreover, the information about retina images which contain in simple cells near S, is invariant with respect to the group B^- since these cells are (locally) parametrized by this group.

We conjecture that the invariancy with respect to the group $G_{-} \simeq \mathbb{R}^2$ of parallel translation is realized on the next level of visual system (where complex cells become important)). It is consistent with a known fact that one of the principal difference between simple and complex cells is that the excitation of simple cells is not invariant with respect to the shift of the contour, but the excitation of complex cells is invariant with respect to such shift, (see for example, M. Hansard, R. Heraud, A differentail model of complex cells,2011).

Note that in model of Petitot-Citti-Sarti, "points" of retina are pinwheels. In model by Bressloff-Cowan, "points" are hypercolumns, which are associated with a pair of pinwheels (S, N). An image with low level of spatial frequency, is detected by a system of columns near the south pole S, which is (locally) parametrized by points of the group $B^+ \simeq Sim(E^2)$. In other words, "points" with low spatial frequency corresponds to stability subgroups $B^$ associated with the north "pinwheel N of the hypercolumn sphere. Similarly, "points" with high spatial frequency is detected by neighborhood of the north pinwheel N of the hypercolumn . This leads to a description of V1 cortex as the Tits model of the conformal sphere, where "points" are stability subgroups. The associated Cartan connection gives a canonical B^- -equivariant identification of infinitesimal geometric structure in different points.

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