Neurogeometry of vision

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Lecture III. Conformal geometry of the sphere and Gombrich etc principle. Application to stability problem

I. Conformal Geometry of sphere

1. Models of conformal sphere.

2. Proposition about action of stabilizer of Möbius group on projective planes.

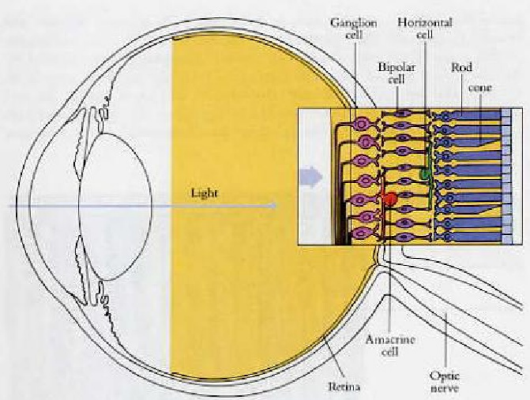
II. Shift of receptive fields and Principle of E. Gombrich.

1. Shift of receptive filed and remappings.

2. Etc Principle by Ernst Gombrich.

III. Conformal identification of images after remapping.

Eye


Ganglion cell
Horizontal cell
Bipolar cell
Rod
cone
Light
Amacrine cell
Retina
Optic nerve

Central projection

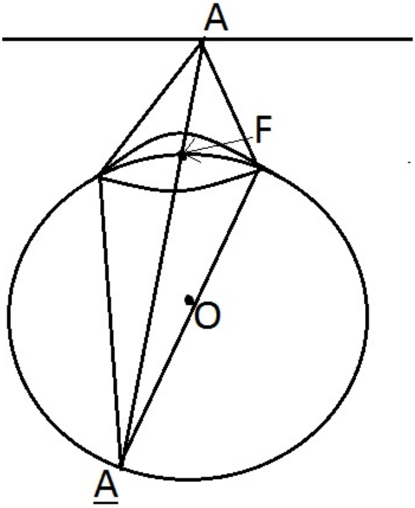


Image A of a point A on the retina

I. Conformal geometry of sphere. Möbius projective model of conformal sphere

If an inertial coordinate system is fixed, the Minkowski space-time $M^{1,3}$ is identified with the vector space

$$M^{1,3} = \mathbb{R}^{1,3} = V = \mathbb{R}e_0 + E^3 \ni X = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 = (x^0, \vec{x})$$

with the Lorentz scalar product $g(X, Y) = -x^0 y^0 + \vec{x} \cdot \vec{y}$.
The light cone at 0 is $V_0 = \{X \in V,\ g(X, X) = 0\}$.
The projectivization $PV_0 = \{[p] = \mathbb{R}p,\ \ p \in V_0\}$ is the conformal 2-sphere.
Three orbits of (connected) Lorentz group $G = SO(V) = SO_{1,3}$:
$V_T = Ge_0 = G/SO_3$ - Lobachevski space,
$V_S = Ge_1 = G/SO_{1,2}$- De Sitter space ,
$V_0 = Gp = G/SE(2)$ - isotropic (light) cone, $SE(2) = SO_2 \cdot \mathbb{R}^2$.
Three G-orbits in $P^3 = PV$: Ball $B^3 = PV_T \simeq V_T$, exterior of ball $PV_S \simeq V_S$, projective quadric = conformal sphere
$S^2 = Q = PV_0 = G/Sim(E^2) = G/(\mathbb{R}^+ \cdot SE(2))$.

Riemann spinor model

$$S^2 = \hat{\mathbb{C}} = P\mathbb{C}^2 = \mathbb{C}P^1$$

Group $\tilde{G} = SL_2(\mathbb{C})$ acts on $\mathbb{C}^2$ and on $\hat{\mathbb{C}} = P(\mathbb{C}^2) = \mathbb{C}P^1$ by fractional-linear transformations. The action is **simply** transitive on triples of points, but not on frames. So Lorentz group is identified with the set of triples.

Gauss decomposition:

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{g}^- + \mathfrak{g}^0 + \mathfrak{g}^+ = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

It integrates to Gauss decomposition of the group $SL_2(\mathbb{C})_{reg} = G^- \cdot G^0 \cdot G^+$, where $G^\pm \simeq \mathbb{C} = \mathbb{R}^2$,$B^\pm = G^0 \cdot G^\pm$ and $G^0 \simeq \mathbb{C}^*$ is the diagonal group. The Lie algebra $\mathfrak{g} = \{X = (b + az + cz^2)\partial_z\}$.

Riemann spinor model

We can also identify the conformal sphere with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \ni z$ with the metric $g(z, z) = \bar{z} \cdot z$ and the fractional-linear action of the unimodular group as the conformal group:

$$SL_2(\mathbb{C}) \ni A \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto Az = \frac{az + b}{cz + d}$$

.

Let

$$Sl_2(\mathbb{C}) = G_- \cdot G_0 \cdot G_+ = \begin{pmatrix} 1 & 0 \\ \mathbb{C} & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}, c, b \in \mathbb{C}$$

be the Gauss decomposition. Then $S^2 = SL_2(\mathbb{C})/B_\pm = SL_2(\mathbb{C})/G_0 \cdot G_+ = SL_2(\mathbb{C})/G_0 \cdot G_-$. Here $B_\pm := G_0 \cdot G_\pm \simeq CO_2 \cdot \mathbb{R}^2$ are the stability subgroup of $z = 0$, resp., ∞.

Tits model

Since any subgroup $H \subset SO(V)$, isomorphic to $Sim(E^2)$, is the stability subgroup $SO(V)_{[p]}$ of unique point $[p] \in S^2$, the conformal sphere is identified with the set of subgroups conjugated (or isomorphic) to $Sim(E^2)$. In other word, in this model (called Tits model) points of the sphere are defined as subgroups of $SO(V)$ conjugated to $Sim(E^2) = CO_2 \cdot \mathbb{R}^2$.

Conformal theory of curves (Fialkov, Sulanke, Sharp, Shelechov)

Let (p, e_1, e_2, q) be an iso-orthogonal basis of V. A curve $\gamma(t)$ can be represented by a curve $r(t) \subset V_0$. We assume that it is regular (s.t. $\dot{r}(t)^2 \neq 0$ and $\rho(t) := \ r(t) \wedge \dot{r}(t) \wedge \ddot{r}(t) \wedge \dddot{r}(t) \neq 0$. Then there is a canonical parameter s (defined up to a shift) and an iso-orthogonal moving frame $p(s), e_1(s), e_)2(s), q(s)$ such that the following conformal Frene equations hold

$$\left\{ \begin{array}{l} \dot{p}(s) = e_1(s) \\ \dot{e}_1(s) = k(s)p(s) + q(s) \\ \dot{q}(s) = k(s)e_1(s) + e_2(s) \\ \dot{e}_2(s) = p(s). \end{array} \right\}.$$

Invariant $k = k(s)$ is called conformal curvature.
Conformal Frenet equations defines the canonical lift of the curve $[\gamma(s)] \subset S^2 = PV_0$ to the Lorentz group G.

Now we return to the projective model of the conformal sphere $S^3 = PV_0$ and its standard realisation as the sphere $S^2 = E^4_{e_0} \cap V_0 \subset E^3_{e_0}$ in the Euclidean hyperplane $E^3_{e_0} = e_0 + E^3 \subset V$.
The intersection of an isotropic line $\mathbb{R}p \subset V_0 \subset V$ with hyperplane $E^3_{e_0} = e_0 + E^3 = \{x_0 = 1\}$ gives a conformal diffeomorphism

$$PV_0 \ni [p] = \mathbb{R}p \Leftrightarrow \mathbb{R}p \cap E^3_{e_0} = e_0 + \vec{x} \in S^2_{e_0} \in e_0 + S^2,$$

$S^2 = \{\vec{x} \in E^3, \vec{x}^2 = 1\}$.
So we may identify the conformal sphere PV_0 as the sphere $S^2_{e_0}$ of the Euclidean space $E^3_{e_0}$ with center at the origin e_0. We call $S^2 = S^2_{e_0}$ the standard realisation of the conformal sphere PV_0 (shortly, the standard sphere). (It is the "celestial sphere "of the observer with world line $\mathbb{R}e_0$.
The Lorentz group $SO(V) = SO_{1,3}$ acts transitively on S^2 as the group of conformal transformations (Möbius group). The stability subgroup $SO(V)_{[p]} = Sim(E^2) = CO_2 \cdot \mathbb{R}^2 = \mathbb{R}^+ \cdot SO_2 \cdot \mathbb{R}^2$. Hence, $S^2 = SO(V)/Sim(E^2)$.

Hyperplanes in V and planes in $E^3_{e_0} = e_0 + E^3$

The hyperplane $n^\perp$ with a normal $n \in V$, $n \neq e_0$ defines a plane $\Pi^n = n^\perp \cap E^3_{e_0}$.
The plane Π^n intersects the sphere $S^2_{e_0} \Leftrightarrow n^2 > 0$ (i.e. n is spacelike) and the intersection $\Pi^n \cap S^2_{e_0}$ is a circle in $S^2_{e_0}$ (conformal geodesic),
The plane Π^n is tangent to sphere $\Leftrightarrow n^2 = 0$ (n is isotropic) and Π^n does not intersect the sphere $\Leftrightarrow n^2 > 0$ (n is timelike).

Theorem.

Let

$$\pi_{\Pi}^{F} : \Pi \to S^2, \ \pi_{\Pi'}^{F} : \Pi' \to S^2$$

be the central projection of two planes $\Pi = \Pi^n$, $\Pi' = \Pi^{n'}$ in $E^3_{e_0}$ which does not intersect the standard sphere $S^2 \subset E^3_{e_0}$ into S^2 with respect to a point $F = [p] \cap E^3_{e_0} \in S^2$. Then there is a Lorentz transformation $\varphi \in SO_{1,3}$, which preserves the point F and transform Π onto Π'.
Moreover, consider points $A \in \Pi, A' = \varphi(A) \in \Pi'$ and denote by $\bar{A} = \pi_{\Pi}^{F}(A) = \ell_{AF} \cap S^2$, $\bar{A}' = \pi_{\Pi'}^{F}(A') = \ell_{A'F} \cap S^2$ their images under cental projection. Then

$$\varphi(\bar{A}) = \varphi(\pi_{\Pi}^{F} A) = \pi_{\varphi\Pi}^{\varphi(F)}(\varphi(A)) = \pi_{\Pi'}^{F}(A') = \bar{A}'$$

In particular, if we identified planes Π, Π' using the transformation φ, then the images of these planes on the sphere are related by the conformal transformation $\varphi|_{S^2}$.

Large Shift of RF and remapping (Jean-Rene Duhamel; Carol L. Colby; Michael E. Goldberg,1992)

In a seminal paper, Duhamel et al. described the shift or RF of many neurons in macaque lateral intraparietal area (LIP). Assume that the RF of a neuron before saccade covers the retina image $\bar{A}$ of a point A and after a saccade the retina image $\bar{A}'$ of another point A'. Then 100 ms before a saccade, the neuron detects stimuli at the locations $\bar{A}'$.
This process constitutes a remapping of the stimulus from the retina coordinates with the initial fixation point to those of the future fixation point . The process is governed by a copy of the motor command (Corollary Discharge)

Remapping of retina images

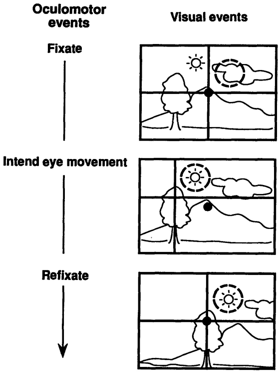
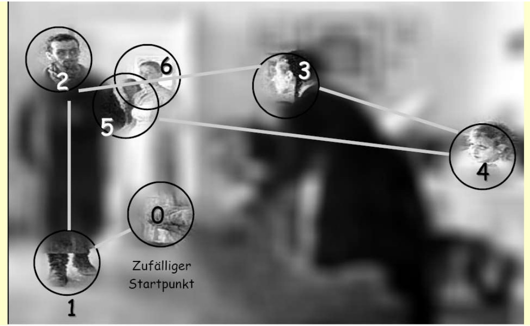


Fig. 1. Remapping of the visual representation in parietal cortex. Each panel represents the visual image at a point in time relative to a sequence of oculomotor events. Receptive field of a parietal neuron, dashed circle; center of current gaze location, solid circle; and coordinates of the cortical representation, cross hairs.

Fixation fields after first 6 saccades



Fixation fields after first 6 saccades

The etcetera principle by art historian Ernst Gombrich

The global pattern in environments such as a forest, beach or street scene enables us to predict more-or-less what we will see, based on the order and redundancy in the scene and on previous experience with that type of environment.
Only a few (3-4) salient stimuli are contained in the trans-saccadic visual memory and update.
(Recall a presentation of a walking person in low cost cartoon).

Problem of stability and conformal Möbius group

We propose a mechanism of remapping by means of conformal transformation of retina, defined by identification of retina images of three salient points (stimuli) before and after remapping.

Conformal identification of retina images before and after saccade (remapping)

Note that the central projection of a plane Π to a sphere S^2 with respect to a point $F \in S^2$ is a projectively invariant notion.
We will use the projective model of the conformal sphere and identify our Euclidean 3-space with the Euclidean subspace $E^3_{e_0}$ of the Minkowski space $V = \mathbb{R}^{1.3}$.
Let $\pi^F_\Pi : \Pi \to S^2$ be the central projection, where $S^2 \subset E^3_{e_0}$ is the eye sphere.
With respect to the coordinates, fixed relatively to the eye sphere S^2, the rotation $R \in SO_3$ of the eye corresponds to the inverse rotation R^{-1} of the external space. It transform the plane Π onto plane $\Pi' := R^{-1}]\Pi$. According to the above theorem, there is a Lorentz transformation φ which fixes F and transforms the plane Π onto the plane Π'. If we identify planes, using φ , then the central projections of $\Pi.\Pi'$ are related by $\pi^F_\Pi = \pi^F_{\Pi'} \circ \varphi$ images $\pi^F()$
Proposition. Let $\pi^\Pi_F : \Pi \to S^2_{e_0}$ be the central projection of a plane and π^Π : $\Pi \to R^{-1}(S^2)$ another central projection obtained by

Conjecture

The brain identify the (stable) backgroup of a picture before and after saccade as a conformal transformation ϕ of the retina defined by condition

$$\phi(\bar{A}_i) = \bar{A}'_i$$

where $\bar{A}_i$ (resp., $\bar{A}'_i$) are retina images of a few salient points of the picture before (resp., after) a saccade.
Since the Möbius group acts simply transitively of triples of points of the sphere, It is sufficient to consider three salient points.
Probably, the brain uses more such points and to correct the claculation of the conformal transformation which maps images of three points d before and after saccades. This conformal transformation identifies the background or environment without change , according to etc principle by Ernst Gombrich.