Introduction to representation theory
Lecture 3
Basic representation theory (beginning)

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(Linear) representations of groups

Let G be a group.

Presentation of G:

An epimorphism $F_S \twoheadrightarrow G$ for a subset S of G, where F_S is the free group generated by S

Representation of G on a vector space V:

An action of G on V by linear transformations; equivalently, a group morphism $G \to GL(V)$.

The same words can be said about some other algebraic structures: associative, associative commutative and Lie algebras.

Examples of group representations

- GL(V) canonically acts on V by linear transformations;
- so does any subgroup of GL(V), eg. SL(V), O(V), SO(V), U(V), SU(V), whenever it is defined;
- V_1 and V_2 are modules over G_1 and G_2, respectively. Then $V_1 \times V_2$ is a module over $G_1 \times G_2$, where the latter is the product of group with the canonical multiplication, neutral element and inverses:

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2'), \quad (g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$$

$$G_1 \times G_2 \ni e = (e, e), \quad g_i, g_i' \in G_i,\ i = 1, 2$$

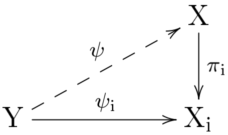
The linear action of $G_1 \times G_2$ on $V_1 \times V_2$ is

$$(g_1, g_2)(v_1, v_2) = (g_1 v_1, g_2 v_2), \quad g_i \in G_i, v_i \in V_i,\ i = 1, 2$$

Product in a category

Let C be a category and $(X_i)_{i \in I}$ be an indexed family of objects, parameterized by an index set I (infinite, in general).

The direct product $X = \prod_{i \in I} X_i$ is an object together with morphisms $\pi_j \colon X \to X_j$ (projections) for all $j \in I$, satisfying the next property: for any family of morphisms $(\psi_i \colon Y \to X_i)_{i \in I}$, there exists a unique morphism $\psi \colon Y \to X$, such that the following diagram is commutative:



If the product exists, it is unique up to canonical isomorphism.

Direct product of sets

Given an $I-$indexed family of sets $(X_i)_{i \in I}$, where I is possibly infinite, the product (the direct product of sets) is the Cartesian product, defined as follows:

$$X = \prod_{i \in I} X_i = \{x \colon I \to \cup_{i \in I} X_i \mid x(i) \in X_i, \forall i \in I\}$$

with the projections $\pi_j \colon X \to X_i$, $\pi_j(x) = x(j)$ for all $j \in I$.

For a family of maps $(\psi_i \colon Y \to X_i)_{i \in I}$, there exists a canonical map $\psi \colon Y \to X$, defined for any $y \in Y$ as follows:

$$\psi(y) \colon I \to \cup_{i \in I} X_i, \quad \psi(y)(i) = \psi_i(y) \in X_i$$

One has $\pi_j(\psi(y)) = \psi(y)(j) = \psi_j(y)$ for all $j \in I$, which is what we need.

Direct product of groups and vector spaces

Given an I−indexed family of groups $(\mathrm{G_i})_{\mathrm{i}\in\mathrm{I}}$ and vector spaces $(\mathrm{V_i})_{\mathrm{i}\in\mathrm{I}}$, the corresponding direct product of sets

$$\tilde{\mathrm{G}} = \prod_{\mathrm{i}\in\mathrm{I}} \mathrm{G_i} = \{\mathrm{g} : \mathrm{I} \to \cup_{\mathrm{i}\in\mathrm{I}} \mathrm{G_i} \,|\, \mathrm{g(i)} \in \mathrm{G_i}\}$$

$$\tilde{\mathrm{V}} = \prod_{\mathrm{i}\in\mathrm{I}} \mathrm{V_i} = \{\mathrm{v} : \mathrm{I} \to \cup_{\mathrm{i}\in\mathrm{I}} \mathrm{V_i} \,|\, \mathrm{v(i)} \in \mathrm{V_i}\}$$

is a group and a vector space, respectively. For groups:

- the multiplication is determined by the property

$$\mathrm{gg'} : \mathrm{i} \mapsto \mathrm{g(i)g'(i)}, \quad \forall \mathrm{g}, \mathrm{g'} \in \tilde{\mathrm{G}}$$

- the neutral element is $\mathrm{i} \mapsto \mathrm{e}$;
- the inverse of $\mathrm{g}^{-1} : \mathrm{i} \mapsto (\mathrm{g(i)})^{-1}$, $\forall \mathrm{i} \in \mathrm{I}$.

Direct product of groups and vector spaces

For vector spaces:

- the $k-$linear combination $cv + c'v'$, where $v, v' \in \tilde{V}$, $c, c' \in k$, acts on the index set I as follows:

$$i \mapsto cv(i) + c'v'(i),$$

- zero in $\tilde{V}$ is the identical zero: $i \mapsto 0$ for all $i \in I$.

The direct sum of vector spaces $\bigoplus_{i \in I} V_i$ is a vector subspace of $\tilde{V}$, consisting of $v : i \mapsto V_i$ with a finite number of non-zero values.

If I is finite, $I = (1, 2, \ldots, n)$, then

- the direct product of groups is the set of $n-$tuples

$$\prod_{i \in I} G_i = \{(g_1, \ldots, g_n) \,|\, g_i \in G_i, i = 1 \ldots n\}$$

  with the componentwise multiplication of $n-$tuples

$$(g_1, \ldots, g_n)(g_1', \ldots, g_n') = (g_1 g_1', \ldots, g_n g_n')$$

  and componentwise inverses

$$(g_1, \ldots, g_n)^{-1} = (g_1^{-1}, \ldots, g_n^{-1})$$

  The neutral element is the diagonal element $(e, \ldots, e)$.
- $n-$tuples are in one-two-one correspondence with maps

$$g : I \mapsto \cup_{i \in I} G_i, \quad g(i) = g_i.$$

- The direct product of vector spaces is the set of n−tuples

$$\prod_{i \in I} V_i = \{(v_1, \dots, v_n) \,|\, v_i \in V_i, i = 1 \dots n\}$$

  with componentwise linear combination of n−tuples for $c, c' \in k$

$$c(v_1, \dots, v_n) + c'(v_1', \dots, v_n') = (cv_1 + c'v_1', \dots, cv_n + c'v_n')$$

  The zero element is the diagonal element $(0, \dots, 0)$.
- n−tuples of vectors are in one-two-one correspondence with maps

$$v : I \mapsto \cup_{i \in I} V_i, \quad v(i) = v_i.$$

- $\bigoplus_{i \in I} V_i = \prod_{i \in I} V_i.$

Examples of group representations

- (Generalization of the product of two representations)

  Let V_i and G_i be indexed families of vector spaces and groups, respectively, parameterized by the same index set I, such that V_i is a G_i module for all $i \in I$.

  Then $\tilde{V} = \prod_{i \in I} V_i$ is a module of $\tilde{G} = \prod_{i \in I} G_i$.

  The representation is defined pointwise, similar to the algebraic structures on the direct products.

- In particular, for a finite set $I = (1, 2, \ldots, n)$ the representation is defined in terms of $n-$tuples

$$(g_1, \ldots, g_n)(v_1, \ldots, v_n) = (g_1 v_1, \ldots, g_n v_n)$$

  where $g_i \in G_i$, $v_i \in V_i$, $i \in I$.

Examples of group representations

- Let $\phi\colon \mathrm{G}_1 \to \mathrm{G}_2$ be a group morphism, V be a G_2-module, then V is a G_1-module:

$$\mathrm{g}_1\mathrm{v} = \phi(\mathrm{g}_1)\mathrm{v}, \quad \mathrm{g}_1 \in \mathrm{G}_1, \mathrm{v} \in \mathrm{V}$$

  Indeed, we take the composition of two morphisms of groups $\mathrm{G}_1 \to \mathrm{G}_2$ and $\mathrm{G}_2 \to \mathrm{GL(V)}$;
- (combination of the previous examples)

  Consider the family $\mathrm{G}_\mathrm{i} = \mathrm{G}$ for $\mathrm{i} \in \mathrm{I}$.

  Take the diagonal embedding $\Delta_\mathrm{I} : \mathrm{G} \hookrightarrow \prod_\mathrm{i} \mathrm{G}$, defined as follows: the image of $\mathrm{g} \in \mathrm{G}$ is the constant map $\mathrm{i} \mapsto \mathrm{g}$.

  If I is finite then $\Delta_\mathrm{I}(\mathrm{g}) = (\mathrm{g}, \mathrm{g}, \ldots)$.

  It is easy to verify that Δ_I is a group morphism, thus $\tilde{\mathrm{V}} = \prod_\mathrm{i} \mathrm{V}_\mathrm{i}$ is a $\mathrm{G}-$module.

Examples of group representations

- Let X be a set and $\mathcal{F}(X)$ be the space of $k-$valued functions on X for a field k.

  $\mathcal{F}(X)$ is a $k-$vector space under pointwise addition and multiplication by a scalar

  $$(c_1 f_1 + c_2 f_2)(x) = c_1 f_1(x) + c_2 f_2(x)$$

  where f_1, f_2 are functions, c_1, c_2 are scalars and x is any element of X.

  Notice that $\mathcal{F}(X)$ is finite-dimensional if and only if X is finite. Indeed, it admits a canonical basis $(\delta_x)_{x \in X}$ parameterized by elements of X:

  $$\delta_x(y) = \begin{cases} 1, & y = x \\ 0, & y \neq x \end{cases}$$

Examples of group representations

$\mathcal{F}(X)$ is a module over the group $\mathrm{Aut}(X)$ of bijections of X:
For any $g \in \mathrm{Aut}(X)$, viewed as a bijection $X \to X$, and $f\colon X \to k$

$$gf = (g^{-1})^*(f) = f \circ g^{-1}\colon X \to k$$

Indeed, this operation respects the product in $\mathrm{Aut}(X)$

$$(g_1g_2)f = f \circ (g_1 \circ g_2)^{-1} = f \circ g_2^{-1} \circ g_1^{-1} = g_1(g_2f)$$

Besides $ef = f \circ \mathrm{Id} = f$ for any $f \in \mathcal{F}(X)$, $g_1, g_2 \in \mathrm{Aut}(X)$.
Therefore $\mathrm{Aut}(X)$ acts on $\mathcal{F}(X)$. Moreover, this action is linear:

$$g(c_1f_1 + c_2f_2)(x) = (c_1f_1 + c_2f_2)(g(x)) = c_1f_1(g(x)) + c_2f_2(g(x))$$

thus $g(c_1f_1 + c_2f_2) = c_1gf_1 + c_2gf_2$.

Examples of group representations

- If X is a G−space, then there is a group morphism $G \to \mathrm{Aut}(X)$. By transitivity, $\mathcal{F}(X)$ is a G−module;
- In particular, the (left) action of G on itself $G \times G \to G$ gives us a representation of G on $\mathcal{F}(G)$, called the left regular representation.
- The right action of G on itself, $(g_1, g_2) \mapsto g_1 g_2^{-1}$, gives us the right regular representation;
- The product $G \times G$ acts on G as follows:

$$(g_1 g_2) g = g_1 g g_2^{-1}, \quad \forall g, g_1, g_2 \in G$$

  Thus $\mathcal{F}(G)$ is a $G \times G$−module;
- Consider the diagonal embedding $\Delta \colon G \hookrightarrow G \times G$; by transitivity, $\mathcal{F}(X)$ is again a G−module (the third way!)

Submodules of a G−module

A submodule of a G−module V is a vector subspace $W \subset V$ closed under the action of G:

$$\forall g \in G, w \in W, \quad gw \in W$$

The zero subspace $\{0\}$ of V and V itself are submodules.
A representation is called irreducible and the module is called simple if there are no submodules but the above two. Otherwise it is reducible.

Given a submodule $W \subset V$, the quotient vector space V/W, defined as the set of equivalence classes under the relation $v \sim v + w$ for all $v \in V$, $w \in W$, together with the canonical structure of a vector space (generalizing the quotient of groups), is a G−module.

Morphisms of modules

A linear map $\varphi\colon V_1 \to V_2$ between two G−modules is called a morphism of modules if it commutes with the G−action, i.e. $\varphi(gv) = g\varphi(v)$ for all $g \in G$, $v \in V$. A morphism of modules is a monomorphism (epimorphism, isomorphism) if the underlying linear map is injective (surjective, bijective), respectively.

- The kernel and the image of φ are G−submodules;
- The composition of two morphisms of G−modules, whenever it is defined, is a morphism of modules;
- The identity map $V \to V$ is a morphism of modules;
- The inverse of any morphism as a linear map, if it is invertible, is again a morphism of G−modules;
- All G−modules together with their morphisms is a category under the composition.

Examples of submodules and morphisms

- For a submodule W of V, the inclusion $\mathrm{W} \hookrightarrow \mathrm{V}$ (the quotient map $\mathrm{V} \twoheadrightarrow \mathrm{V}/\mathrm{W}$) is a monomorphism (an epimorphism) of modules, respectively;
- A vector space over a field k as a k^*-module under the multiplication. Any vector subspace is a submodule and any linear map is a morphism of modules;
- For a $\mathrm{G}-$module V, the subspace of linear functions $\mathrm{V}^* \subset \mathcal{F}(\mathrm{V})$ is a $\mathrm{G}-$submodule (called the dual module);
- If X is a topological space (a smooth manifold) and G acts on X by continuous (smooth) maps, then the subspace of continuous (smooth) functions on X is a $\mathrm{G}-$submodule.
- Given a family of modules $(\mathrm{G_i}, \mathrm{V_i})_{\mathrm{i}\in\mathrm{I}}$, the action of $\tilde{\mathrm{G}} = \prod_{\mathrm{i}\in\mathrm{I}} \mathrm{G_i}$ preserves $\bigoplus_{\mathrm{i}\in\mathrm{I}} \mathrm{V_i} \subset \tilde{\mathrm{V}} = \prod_{\mathrm{i}\in\mathrm{I}} \mathrm{V_i}$, hence it is a $\tilde{\mathrm{G}}-$submodule;

Trivial representation and invariant vectors

Given any group G and a vector space V, there always exists trivial representation of G on V:

$$\mathrm{G} \ni \mathrm{g} \mapsto \mathrm{Id} \in \mathrm{GL}(\mathrm{V})$$

For any G−module, the set of invariant vectors

$$\{\mathrm{v} \in \mathrm{V} \,|\, \mathrm{gv} = \mathrm{v},\ \forall \mathrm{g} \in \mathrm{G}\}$$

is a G−submodule, which is a trivial module itself.

- Consider the following action of U(1) on $\mathbb{R}^3$:

  $$\mathrm{U}(1) \ni \exp(\mathrm{i}\theta) \colon (\mathrm{x}, \mathrm{y}, \mathrm{z}) \mapsto (\cos\theta \mathrm{x} - \sin\theta \mathrm{y}, \sin\theta \mathrm{x} + \cos\theta \mathrm{y}, \mathrm{z})$$

  Then z−axis is the submodule of invariant vectors.

Schur's lemma. Part 1

If V_1 and V_2 are two finite-dimensional irreducible representations of a group G and φ is a morphism of modules (also called a $G-$linear map or a linear map over G) - a linear transformation from V_1 to V_2 that commutes with the action of the group, then either φ is invertible, or $\varphi = 0$.

The proof is simple: the kernel and the image of φ are $G-$submodules of V_1 and V_2, respectively. Since both modules are simple, each of these submodules is either zero or coincides with the whole module. There are two options:

1. $\mathrm{Im}\varphi = 0$, then $\varphi = 0$;
2. $\mathrm{Im}\varphi = V_2$, then $\mathrm{Ker}\varphi$ must be zero, hence φ is an isomorphism of the modules.

Schur's lemma. Part 1. Example

Here we use the notations $\mathrm{Hom}_G(V_1, V_2)$ and $\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V)$ for linear maps over G (homomorphisms and endomorphisms of modules, respectively).

Consider the algebra of quaternions $\mathbb{H}$: it has a vector basis $(1, i, j, k)$, consisting of anti-commuting complex units, i.e.

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = -ji,\ ik = -ki,\ jk = -kj$$

with an additional relation

$$ijk = -1$$

The multiplication on $\mathbb{H}$ is obtained by extending of the above relations by linearity.

Schur's lemma. Part 1. Example

For any $x = x_0 + x_1 i + x_2 j + x_3 k$, define the real and imaginary parts of x

$$\mathrm{Re}(x) = x_0, \quad \mathrm{Im}(x) = x_1 i + x_2 j + x_3 k$$

the conjugate

$$\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$$

which satisfies $\overline{xy} = \bar{y}\bar{x}$ for all pairs of quaternions, and the absolute value

$$|x|^2 = x\bar{x} = \bar{x}x = \sum_{r=0}^{3} x_r^2$$

Then for any non-zero x

$$x^{-1} = \frac{\bar{x}}{|x|^2}$$

Schur's lemma. Part 1. Example

$\mathbb{H}$ is non-commutative, however, $\mathrm{Re}(xy) = \mathrm{Re}(yx)$ for any two quaternions x and y.

One has $|xy| = |x||y|$ and $|1| = 1$, therefore

1. the set of unitary quaternions $\mathrm{Sp}(1)$ is a subgroup of $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$;
2. $\mathrm{Sp}(1)$ acts on $\mathbb{H}$ on the left by orthogonal transformations, hence $\mathbb{H} \simeq \mathbb{R}^4$ is an $\mathrm{Sp}(1)-$module;
3. consider $\mathbb{H}$ as a complex vector space, where the multiplication by complex numbers is given by the action of $\mathbb{C} = \{x_0 + x_1 i\} \subset \mathbb{H}$ on the right, then $\mathrm{Sp}(1)$ becomes a complex subgroup of $\mathrm{GL}_2(\mathbb{C})$;
4. 2 and 3 implies that $\mathrm{Sp}(1) \subset \mathrm{U}(2) = \mathrm{O}(4) \cap \mathrm{GL}_2(\mathbb{C})$. One can show that $\mathrm{Sp}(1) = \mathrm{SU}(2)$.

Schur's lemma. Part 1. Example

Explanation. The real scalar product on $\mathbb{H} \simeq \mathbb{R}^4$

$$(x, y) = \mathrm{Re}(\bar{x}y), \quad \forall x, y \in H$$

extending the norm $|x|^2$ to all pairs of vectors, makes $\mathbb{H}$ into a Euclidean vector space. The right action by imaginary quaternions is skew-adjoint with respect to this scalar product: for any $z \in H$, such that $\mathrm{Re}(z) = 0$ and thus $\bar{z} = -z$ one has

$$(x, yz) = \mathrm{Re}(\bar{x}yz) = \mathrm{Re}(z\bar{x}y) = -\mathrm{Re}(\overline{xz}y) = -(xz, y)$$

In general, a complex vector space together with a real scalar product is Hermitian if the multiplication on the complex unit is skew-adjoint. Then the corresponding Hermitian product $\langle , \rangle$, with the property $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$, is defined as follows:

$$\langle v_1, v_2 \rangle = (v_1, v_2) + \sqrt{-1}(Iv_1, v_2), \quad v_1, v_2 \in V$$

Schur's lemma. Part 1. Example

On the other hand, for any $x \in \mathbb{H}^*$ and $y \in H$,

$$\mathrm{Re}(xyx^{-1}) = \mathrm{Re}(x^{-1}xy) = \mathrm{Re}(y)$$

therefore the subspace of imaginary quaternions

$$\{y \in \mathbb{H} \,|\, \mathrm{Re}(y) = 0\} \simeq \mathbb{R}^3$$

is stable under conjugation $y \mapsto xyx^{-1}$, thus it is an $Sp(1)-$submodule of $\mathbb{H}$.

Taking into account that $Sp(1)-$action preserves the scalar product, we obtain a group morphism $Sp(1) \to O(3)$. It induces an epimorphism $Sp(1) \twoheadrightarrow SO(3)$, the kernel of which is $\{\pm 1\}$.

We got two representations of $Sp(1) \simeq SU(2)$ of different dimensions, both are irreducible, thus there are no $Sp(1)-$linear maps between them, except the trivial one $(= 0)$.

Schur's lemma. Part 2

2. If V is an irreducible finite-dimensional G$-$module over an algebraically closed field k and φ is an morphism of modules $V \to V$, then it is a scalar multiple of the identity.

The proof: since k is algebraically closed (eg. $k = \mathbb{C}$), there exists an eigenvector $v \neq 0$ of φ corresponding to an eigenvalue $\lambda \in k$, $\varphi(v) = \lambda v$. The $\lambda-$eigenspace V_λ of φ is a G$-$submodule, since for any $v \in V_\lambda$ and any $g \in G$, one has $\varphi(gv) = g\varphi(v) = \lambda gv$, therefore $V_\lambda = V$ and φ acts by multiplication on λ.

Remark: For a more general field k the space of G$-$linear endomorphisms is an algebra over k with division.

Schur's lemma for $k = \mathbb{R}$

There are only three division algebras over the field of real numbers: $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. The first and the second algebras are commutative (fields), the third one is not. All three division algebras can be obtained via Schur's lemma.

- $G = GL_n(\mathbb{R})$ with the canonical (or standard) representation on $\mathbb{R}^n$, $\mathrm{End}_G(\mathbb{R}^n) = \mathbb{R}$;
- $G = GL_n(\mathbb{C})$ with the canonical representation on $\mathbb{C}^n$, then $\mathrm{End}_G(\mathbb{C}^n) = \mathbb{C}$;
- $G = GL_n(\mathbb{H})$, the group of invertible matrices whose entries are quaternions, with the standard representation on $\mathbb{H}^n$ by left action on vector columns. Then $\mathrm{End}_G(\mathbb{H}^n) = \mathbb{H}$, where $\mathbb{H}$ is acting on vectors on the right.