Introduction to representation theory
Lecture 1
Symmetry in Nature, Algebra, and Geometry

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Symmetry: definition and examples

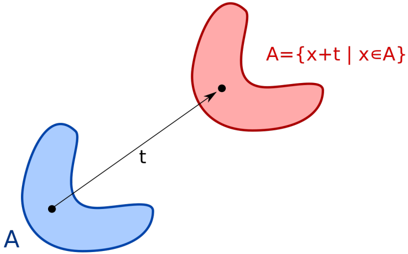
Symmetry - correspondence, immutability (invariance), manifested in any changes, transformations (for example: position, energy, information, other).

Examples.

- Translation in time and space
- Rotations about the axis
- Mirror reflections about an axis (plane, ...)
- Motions, inversions, more general isometries, projective and conformal transformations, ...
- Gauge symmetries
- Supersymmetries
- ...

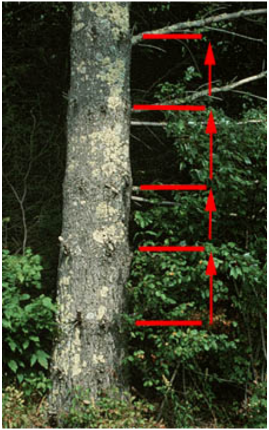
Translation symmetry

Translational symmetry is a type of symmetry in which the properties of the system under consideration do not change when shifted by a certain vector, which is called the translation vector.



Translation symmetry in one direction

- Weymouth pine (white eastern or northern white pine, lat. Pinus strobus) each year forms a new ring of branches, with some variations depending on the year



Translation symmetry in one direction

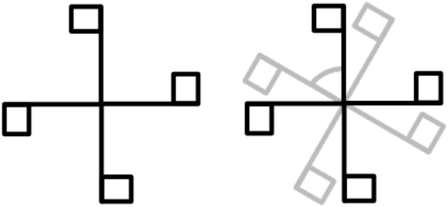
- Scale of the Buff striped keelback

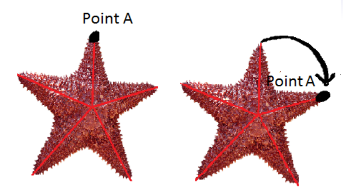
- Rails

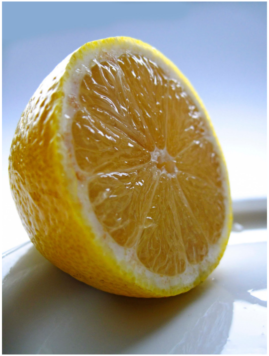
Translation symmetries in two directions

Rotation symmetry

Rotational symmetry or radial symmetry is a property of a geometric object when, being rotated by a certain angle about a certain axis of rotation, it will be aligned with its original position.




Point A
Point A

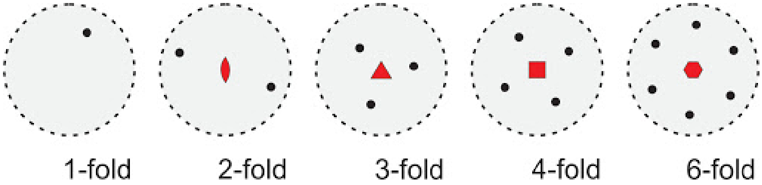


- Great Pyramid of Giza (also known as the Pyramid of Khufu or the Pyramid of Cheops). The length of the sides of the base of the pyramid: south - 230.454 m; north - 230.253 m; west - 230.357 m; east - 230.394 m

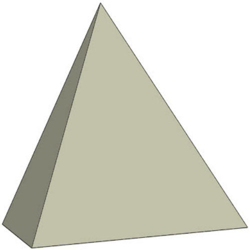
- Surface of revolution

- can combine rotation and translation symmetries

The smallest (but not equal to zero) angle $\alpha = \frac{360°}{\text{n}}$ of rotation around the axis of rotation, if is exists, is called the (elementary) angle of rotation, and n is the order of the rotational axis.

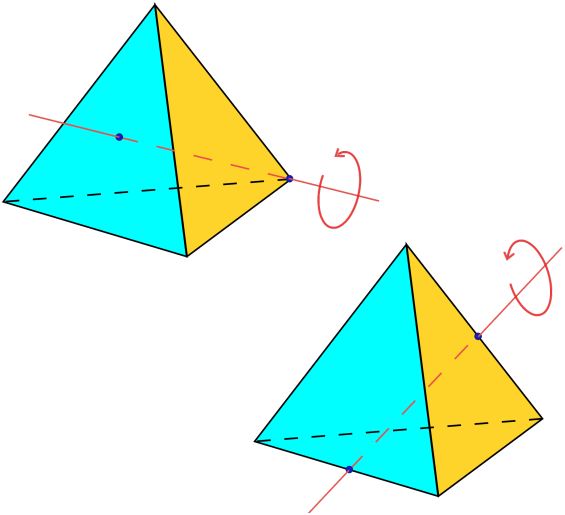


Rotational (or chiral) tetrahedral symmetry

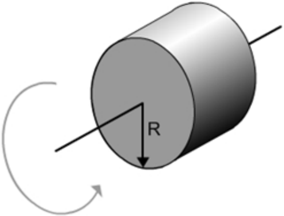


There are:

- three orthogonal 2-fold rotation axes, each goes through one of the vertices and the center of the opposite face
- four 3-fold axes, centered between the three orthogonal directions (equivalently, which go through the midpoints of the opposite edges).

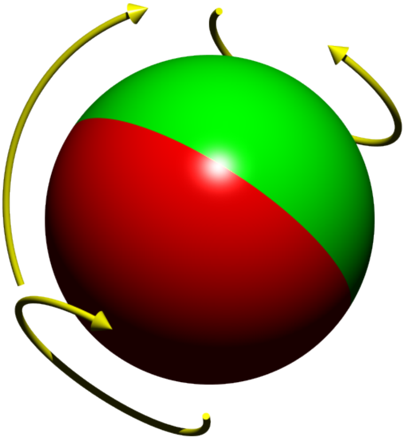


- The cylinder can be rotated at an arbitrary angle around the central axis

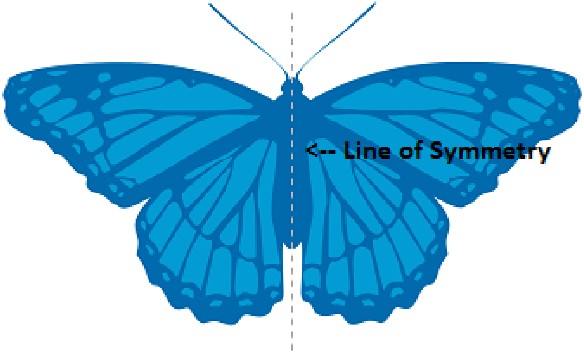


- although some cylinders have broken symmetries

Rotational symmetry of a sphere



Mirror symmetry or reflection

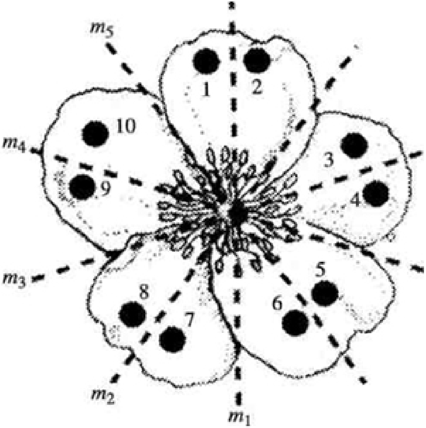


This is an example of bilateral symmetry



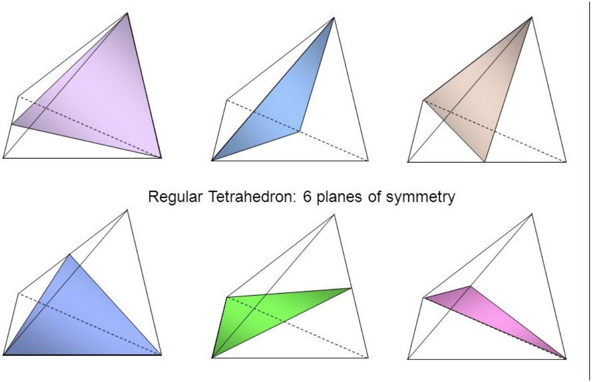

Floral symmetry

- Д.Б. Гелашвили, Е.В. Чупрунов, М.О. Марычев, Н.В. Сомов, А.И. Широков, А.А. Нижегородцев. Приложение теории групп к описанию псевдосимметрии биологических объектов. Журнал общей биологии. Том 71, № 6. Ноябрь-декабрь, 2010, стр. 497-513.
- Елена Бадьева. Посчитанные отражения. Элементы большой науки (https://elementy.ru/genbio/synopsis/326/Prilozhenie_teorii_grupp_k_opisaniyu_psevdosimmetrii_biologicheskikh_obektov)

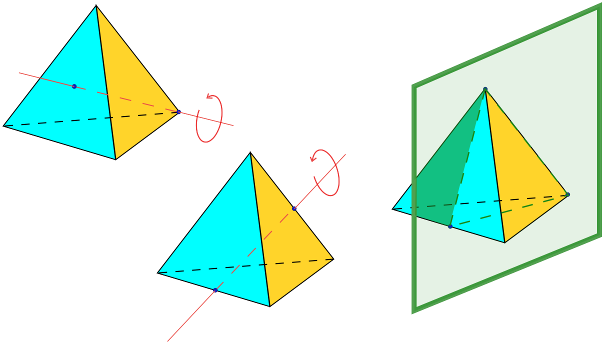


Scheme of a 5-petal actinomorphic flower: m1 ... m5 - planes of symmetry; 1 ... 10 - auxiliary points, coinciding thanks to the rotational symmetry. Figure from the article above.

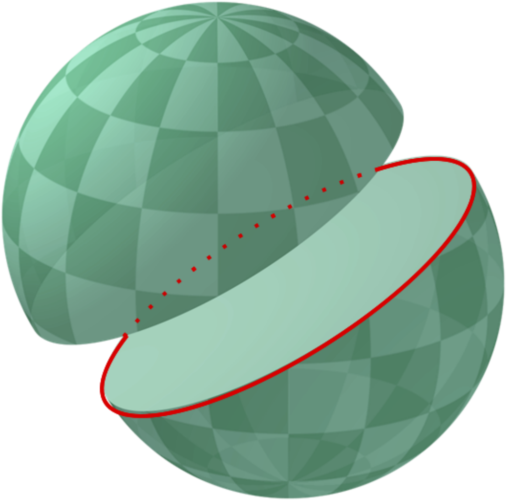
Mirror tetrahedral symmetries



The tetrahedron has six planes of symmetry. Each plane contains the center of the tetrahedron and one edge (and so bisects the opposite edge). Equivalently, each plane of symmetry contains one edge and is orthogonal to the opposite edge.



Mirror symmetries of sphere



- Isometries are transformations that preserve lengths (distances between points)
  - In "usual"Euclidean 3D space, a combination of translations and rotations (plus mirror symmetries if we don't care about orientation)
  - For the sphere - rotations (plus mirror symmetries)
  - For the cylinder - a combination of shifts along the axis of symmetry and rotations around the axis of symmetry (plus mirror symmetries)
- When the symmetries of a space are known, the most symmetric objects of this space (may) solve optimization problems
  - In three-dimensional space, the shortest distance is a straight line segment,
  - while a circle gives the best ratio between perimeter and area.

Symmetries and psychology

Max Wertheimer (1880 - 1943) is one of the founders of Gestalt psychology. He was showing (to the subjects) primitive symmetrical figures in which separate sections were cut out. When the experimenter tried to patch a hole in the circle with a piece cut out of a square, the children began to worry and protest. They wanted the shape of the figure to look perfect.

Thus, our perception is directed towards correctness and symmetry and therefore tends to impose these properties on the observed objects.

This appears to be a general principle, stemming in part from the limited ability of our brains to process information. The bandwidth of our short-term memory - something around 16 bit/s; if information arrives at a slower rate, then we feel bored, and if at a higher rate, then we are overloaded. In complex structured stimuli (patterns), we try to find an ordering that would allow us to distinguish larger elements ("supersigns") in them and thus deal with less information.

Eibl-Eibesfeldt, Irenaus. The Biological Foundation of Aesthetics. Chapter 2 in "Beauty and the Brain. Biological Aspects of Aesthetics".— Birkhäuser Basel, 1988.

What makes a face attractive?

First, symmetry: we prefer it to asymmetry. What makes a face attractive?

This applies to both male and female faces, according to David Perrett, head of the Perception Research Lab at the University of St Andrews, Scotland. Experiments have shown that in all cultures, men and women like symmetrical faces more than asymmetric ones.

Moreover, the preference for symmetry plays an important role in the choice of mate, not only in humans and apes, but also in birds, even insects.

Perrette believes that ill health and adverse environmental influences can lead to asymmetry, and the degree of facial symmetry can serve as an indicator of the genome's ability to resist disease and maintain normal development in adverse conditions. In addition, the stability of developmental mechanisms is largely inherited. Therefore, symmetry, at least face, is beautiful not only for formal reasons, but also due to the fact that she speaks about the health of a potential partner and the health of his possible offspring.

Eric Kandel. The Age of Insight: The Quest to Understand the Unconscious in Art, Mind, and Brain, from Vienna 1900 to the Present. Random House; 1st edition (March 27, 2012)

Side effects

- Apophenia is the tendency to perceive meaningful connections between meaningless or unrelated things. The term (German: Apophanie) was coined by psychiatrist Klaus Conrad in his 1958 publication on the beginning stages of schizophrenia.

- conspiracy theories

Amorphous and crystalline solids

In solids, atoms can be placed in space in two ways:

1. The disordered arrangement of atoms when they do not occupy a certain place relative to each other. Such bodies are called amorphous.
2. An ordered arrangement of atoms, when atoms occupy quite definite places in space. Such substances are called crystalline.

Amorphous substances have formal features of solids, i.e. they are able to maintain a constant volume and shape. However, they do not have a specific melting point or crystallization point.

Primitive cell. Bravais lattice

A primitive cell is the minimal imaginary volume of a crystal, parallel transfers (translations) of which in three dimensions make it possible to build a crystal lattice as a whole.
A conventional cell is the smallest unit cell whose axes follow the symmetry axes of the crystal structure.
3 rules of choice (Bravais)

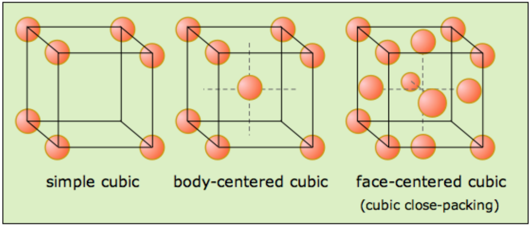
1. The symmetry of the conventional cell must correspond to the symmetry of the crystal;
2. A conventional cell must have the maximum number of equal edges and equal corners
3. Provided that the first two rules are met, the conventional cell should have a minimum volume.

Cubic crystal system

1. The primitive cubic system (cP) consists of one lattice point on each corner of the cube. Each atom at a lattice point is then shared equally between eight adjacent cubes, and the unit cell therefore contains in total one atom. Example: Polonium
2. The body-centered cubic system (cI) has one lattice point in the center of the unit cell in addition to the eight corner points. It has a net total of 2 lattice points per unit cell. Examples: in certain temperature ranges iron, chromium, vanadium, tungsten, molybdenum, and other metals.

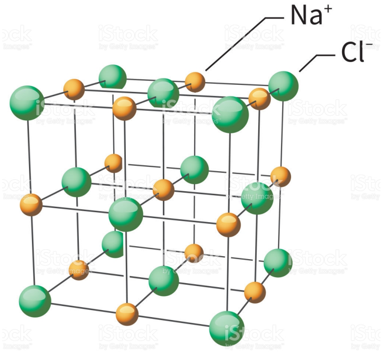
Cubic crystal system

1. The face-centered cubic system (cF) has lattice points on the faces of the cube, that each gives exactly one half contribution, in addition to the corner lattice points, giving a total of 4 lattice points per unit cell. Each sphere in a cF lattice has coordination number 12. Coordination number is the number of nearest neighbors of a central atom in the structure. Examples: iron, aluminum, copper, nickel, lead and other metals.



Crystal structure of Sodium chloride

commonly known as salt. NaCl is a crystal structure with a face centered cubic Bravais lattice and two atoms in the basis. There are 4 Na+ atoms (purple) and 4 Cl- atoms (green) in the conventional unit cell. One Na atom comes from the 8 corners and 3 from the six faces. One Cl atom is in the center and 3 come from the twelve edges.



Penrose tiling

In 1974, Roger Penrose (born August 8, 1931) invented a way to pave an endless plane in a never-repeating pattern using two simple tiles.

Penrose tiling

There are three types of Penrose tiling. All three types, like any aperiodic tiling, have the following properties: the Penrose tiling, being non-periodic, have no translational symmetry – the pattern cannot be shifted to match itself over the entire plane. However, any bounded region, no matter how large, will be repeated an infinite number of times within the tiling. Despite their lack of translational symmetry, Penrose tiling may have both reflection symmetry and fivefold rotational symmetry.

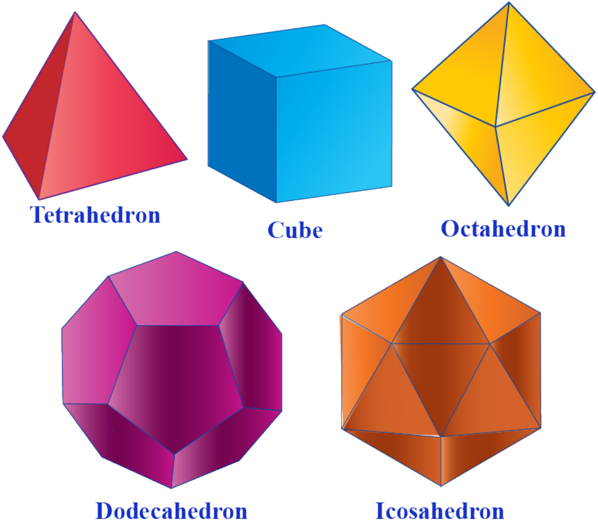
On April 8, 1982, Israeli physicist and chemist Dan Shechtman (Nobel Prize in Chemistry, 2011) in experiments on electron diffraction on a rapidly cooled alloy obtained the first quasicrystalline alloy (today known as "shechtmanite") - a structure with 5th order symmetry.

Platonic solid

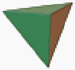
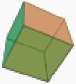
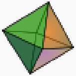
In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent (identical in shape and size), regular (all angles equal and all sides equal), polygonal faces with the same number of faces meeting at each vertex.

Five solids meet these criteria:

The five Platonic solids

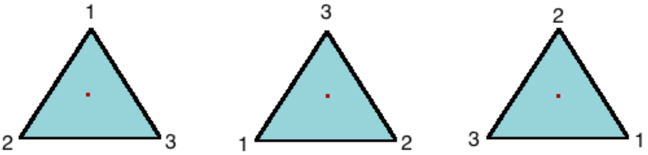


A Platonic solid is uniquely characterized by its Schläfli symbol $\{p, q\}$: its faces are p-gons, and each vertex is surrounded by q faces (the vertex figure is a q-gon).

Polyhedron		Vertices	Edges	Faces	Schläfli symbol	Vertex configuration
tetrahedron		4	6	4	{3, 3}	3.3.3
cube / hexahedron		8	12	6	{4, 3}	4.4.4
octahedron		6	12	8	{3, 4}	3.3.3.3
dodecahedron		20	30	12	{5, 3}	5.5.5
icosahedron		12	30	20	{3, 5}	3.3.3.3.3

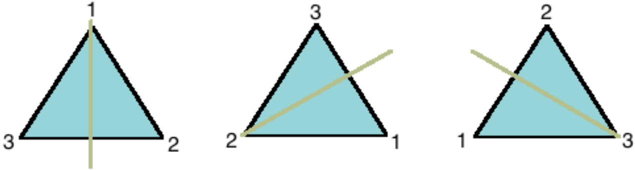
Equilateral triangle symmetry

Counterclockwise rotational symmetries

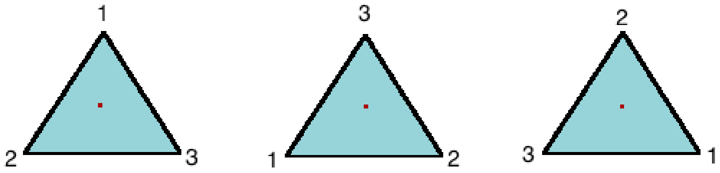


Rotation by 0 degrees. Rotation by 120 degrees. Rotation by 240 degrees.

Reflection symmetries



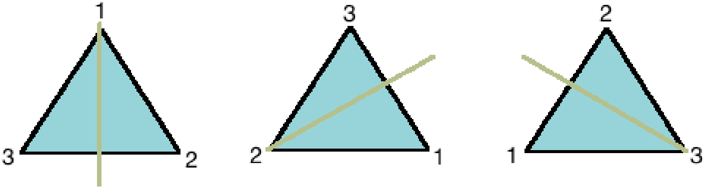
Counterclockwise rotational symmetries of the Equilateral Triangle



Rotation by 0 degrees. Rotation by 120 degrees. Rotation by 240 degrees.

	id	R	R^2
id	id	R	R^2
R	R	R^2	id
R^2	R^2	id	R

Mirror symmetries along an axis symmetry of the Equilateral Triangle



	S_1	S_2	S_3
S_1	id	R	R^2
S_2	R^2	id	R
S_3	R	R^2	id

Cayley (or multiplication) table of symmetries for the Equilateral Triangle

	id	R	R^2	S_1	S_2	S_3
id	id	R	R^2	S_1	S_2	S_3
R	R	R^2	id	S_3	S_1	S_2
R^2	R^2	id	R	S_2	S_3	S_1
S_1	S_1	S_2	S_3	id	R	R^2
S_2	S_2	S_3	S_1	R^2	id	R
S_3	S_3	S_1	S_2	R	R^2	id

The concept of a group

A group (in algebra) - is a set G together with an operation, which associates to each pair of elements g_1, g_2 their product $g_1 \circ g_2$, satisfying the following properties :

- the product is associative, i.e. $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$;
- there exists the neutral element (identity) e, such that for any g one has $g \circ e = e \circ g = g$;
- for each g there exists the inverse g^{-1}, such that $g \circ g^{-1} = g^{-1} \circ g = e$.

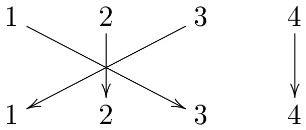
Examples of groups:

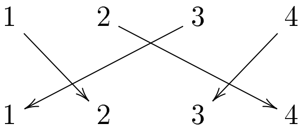
- non-zero real or rational with respect to the usual multiplication;
- symmetries of an object with respect to the composition.

Examples of symmetry groups

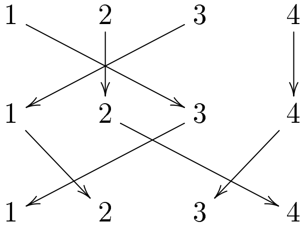
- Permutations of n elements, denoted as $\mathbb{S}_{\mathrm{n}}$.

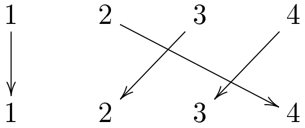
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$





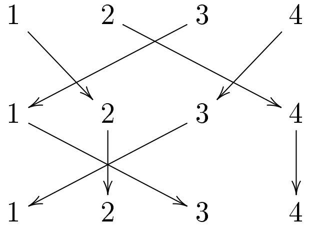
To get the product of two permutations $\sigma \circ \tau$, perform first τ, then σ:

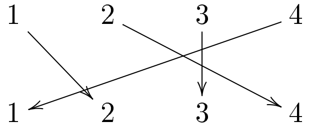




The result is $\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$

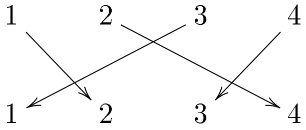
To get the product of two permutations $\tau \circ \sigma$, perform first σ, then τ:

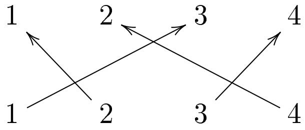




The result is $\tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \neq \sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \qquad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

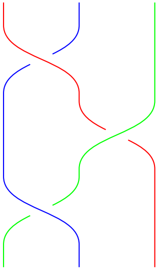




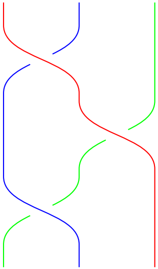
$$\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = \mathrm{id} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

The number of elements in $\mathbb{S}_\mathrm{n}$ is $\mathrm{n}! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot \mathrm{n}$;

Braid group (also known as the Artin braid group)

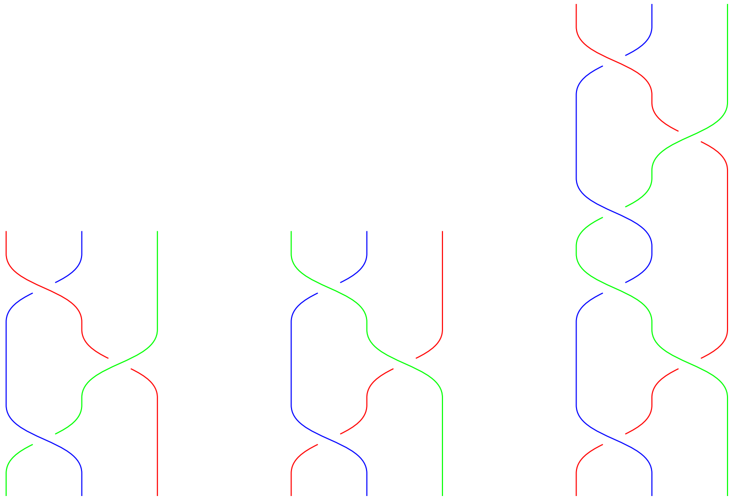


$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

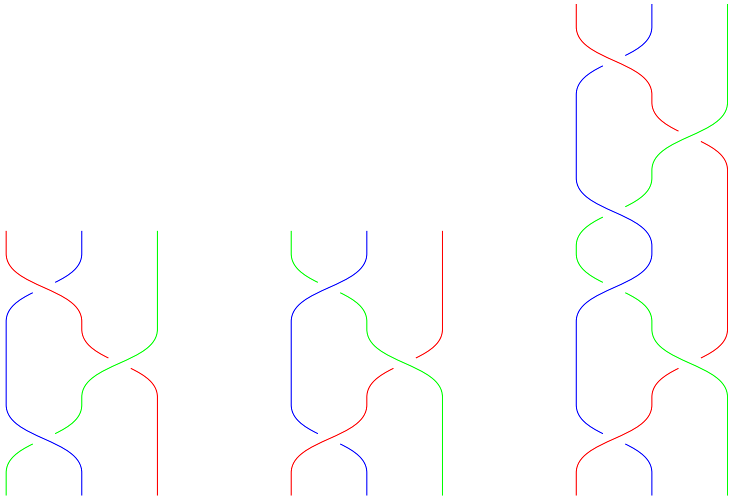


$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Multiplication in the braid group B_3 (3-braids)



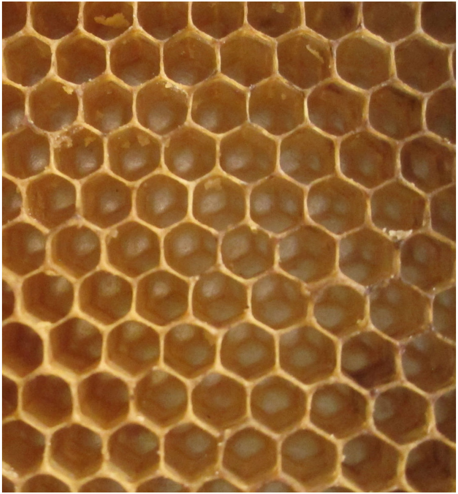
The inverse element



Examples of symmetry groups

- Symmetries of the Equilateral Triangle (the same as $\mathbb{S}_3$);
- symmetry of the tetrahedron (the same as $\mathbb{S}_4$);
- symmetries of other regular polygons and polyhedra;
- groups, consisting of point symmetries (leave at least one point fixed);
- 32 crystallographic point groups (allow axes of rotation only of order 1, 2, 3, 4 and 6);
- 230 crystallographic groups.

Why does the honeycomb have the correct hexagonal structure?



Groups in algebra

- Galois theory. Appeared from attempts to find a formula in radicals for the roots of an equation of arbitrary degree.

The Babylonians were able to solve quadratic equations in the second millennium BC.

Cubic equations - Scipion del Ferro (1465-1526), Niccolo Tartaglia (1500-1557) and Gerolamo Cardano (1501-1576).

Equation of the fourth degree - Lodovico (Luigi) Ferrari (1522-1565) and Gerolamo Cardano.

4 - this is the highest degree of the equation for which a general formula for the solution can be established (the Abel-Ruffini theorem, 1824).

The proof by Paolo Ruffini (1765-1822), published in 1799, was inaccurate and took about 500 pages. The complete proof belongs to Niels Henrik Abel (1802-1829).

The proofs are based on Lagrange's ideas related to permutations of the roots of the equation.

Evariste Galois (1811-1832) proved that for equations of degree 5 and higher the solution "in radicals"is impossible. He introduced abstract criteria for finding roots in terms of groups (now the theory of groups and Galois extensions). Thanks to his ideas, mathematics turned from a science of computation to a science of structures.

Ian Stewart. Why Beauty Is Truth: A History of Symmetry (2007)

Lie groups

A group, such that the set G inherits a smooth structure, i.e. is a manifold, and all operations are smooth. Example: rotation of sphere, rigid motions in the Euclidean space, etc.

Sophus Lie, 1842 – 1899

- "Erlangen program"by Felix Klein (1849-1925) - branches of geometry are classified by different groups of space transformations More precisely, every geometry (Riemannian, conformal, projective, ...) has a "flat model"(homogeneous space), which admits the maximum symmetry group. The obstruction to being "flat"is the curvature, which also reduces the symmetry group;
- symmetries of differential equations (as soon as we know one solution, we can produce many by use of the symmetry group; we may also impose the invariance property with respect to all or some symmetries - the equation for invariant solutions may turn out to be simpler than the original one);
- many special functions have group theory meaning (eg. Fourier series or Schur polynomials);

Symmetries and conservation laws

Noether's theorem (1918) states that each continuous symmetry of a physical system corresponds to a certain conservation law

Amalie Emmy Noether (March 23, 1882, Erlangen, Germany - April 14, 1935, Bryn Mawr, Pennsylvania, USA)

uniformity of time (translation invariance in time)	law of energy conservation
homogeneity of space (invariance with respect to space translations)	momentum conservation law
isotropy of space (invariance with respect to rotations)	angular momentum conservation law

Symmetry breaking

- Violation of global translational invariance (inhomogeneity and finiteness of space).
- Anisotropy, crystal defects.
- Spontaneous symmetry breaking in physics (for example, in elementary particle physics).
- Unpaired internal organs (heart, liver, spleen, ...)
- Asymmetric functions of organs with mirror symmetry (eg cerebral hemispheres, see research by Roger Sperry and Michael Gazzaniga on split-brain syndrome in patients with a severed corpus callosum).

Michael Gazzaniga. Who's In Charge? Free Will And The Science Of The Brain. Ecco; Reprint edition (November 15, 2011).