# On Sectional Curvature Operator of 3-dimensional Locally Homogeneous Lorentzian Manifolds

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Let $(M, g)$ — $n$-dimensional manifold with (pseudo)Riemannian metric.

## The Riemann tensor

$$R(X,Y)Z = [\nabla_Y, \nabla_X]Z + \nabla_{[X,Y]}Z.$$

## The Ricci tensor

$$r(X,Y) = \mathrm{tr}(V \to R(X,V)Y).$$

## The Ricci operator

$$r(X,Y) = g(\rho(X), Y)$$

## Sectional curvature operator $\mathcal{K}$

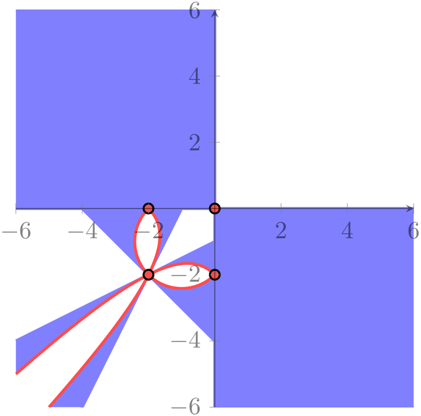
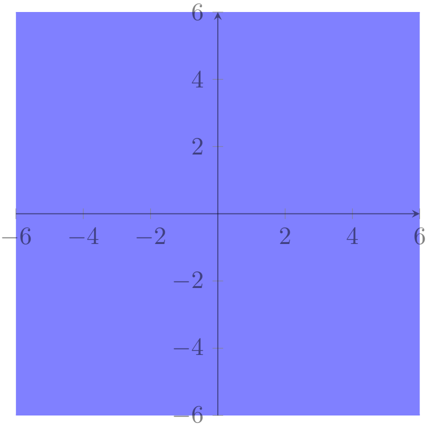
$$\langle X \wedge Y, \mathcal{K}(T \wedge V)\rangle_x = R_x(X,Y,T,V),$$

where $\langle X_1 \wedge X_2, Y_1 \wedge Y_2\rangle_x = \det\left(g_x(X_i, Y_j)\right)$.

# Segre types

| Segre type | Jordan form |
|---|---|
| $\{111\}$ | $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ |
| $\{1z\bar{z}\}$ | $\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \; c \neq 0$ |
| $\{21\}$ | $\begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$ |
| $\{3\}$ | $\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$ |

# Possible Ricci eigenvalues. Segre type $\{111\}$

| Riemannian metric[1] $(\rho_1, \rho_2, -2)$ | Lorentzian metric[2] $(\rho_1, \rho_2, -2)$ |
|---|---|
|  |  |





[1] Kowalski O., Nikcevic S. On Ricci eigenvalues of locally homogeneous Riemann 3-manifolds // Geom. Dedicata. — 1996. — No. 1. — P. 65–72.

[2] Calvaruso G., Kowalski O. On the Ricci operator of locally homogeneous Lorentzian 3-manifolds // Cent. Eur. J. Math. — 2009. — V. 7(1). — P. 124–139.

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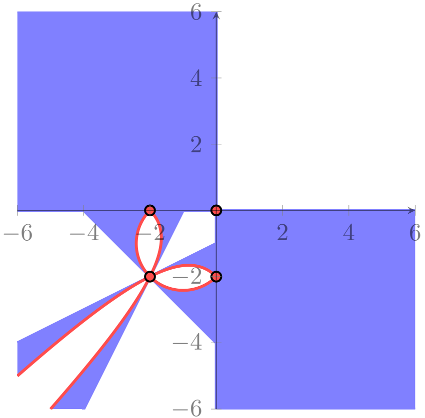
|  | Changing $\rho_1 \to -\rho_1$, $\rho_2 \to -\rho_2$, $\rho_3 \to -\rho_3$ |



[1] Kowalski O., Nikcevic S. On Ricci eigenvalues of locally homogeneous Riemann 3-manifolds // Geom. Dedicata. — 1996. — No. 1. — P. 65–72.

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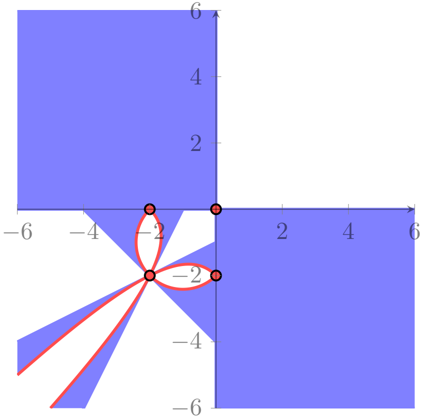
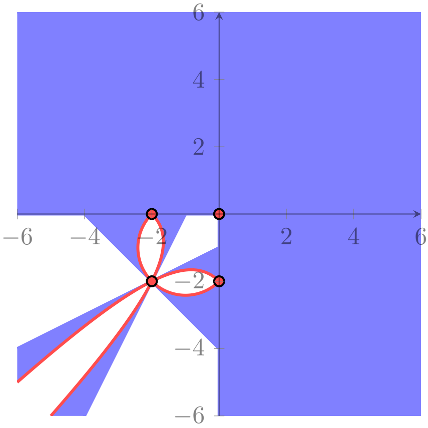
# Possible Ricci eigenvalues. Segre type $\{111\}$

| Riemannian metric[1] $(\rho_1, \rho_2, -2)$ | Lorentzian metric[2] $(-\rho_1, -\rho_2, 2)$ |
| --- | --- |
|  |  |

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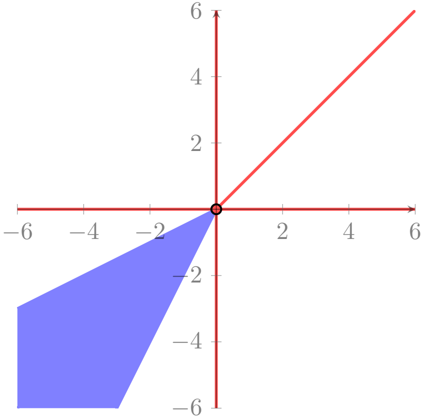
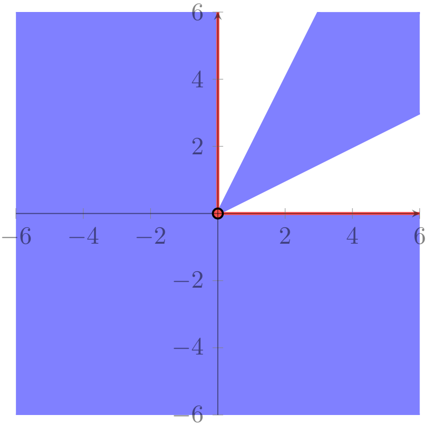
# Possible Ricci eigenvalues. Segre type $\{111\}$

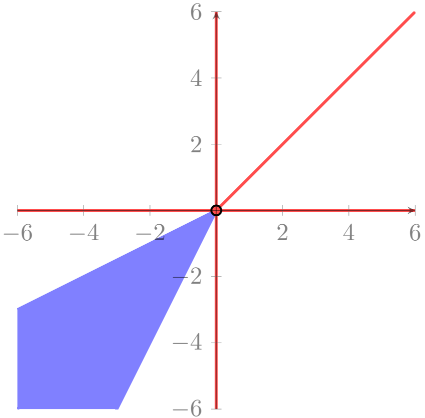
| Riemannian metric[3] $(\rho_1, \rho_2, 0)$ | Lorentzian metric[4] $(\rho_1, \rho_2, 0)$ |
|---|---|




[3] Kowalski O., Nikcevic S. On Ricci eigenvalues of locally homogeneous Riemann 3-manifolds // Geom. Dedicata. — 1996. — No. 1. — P. 65–72.

[4] Calvaruso G., Kowalski O. On the Ricci operator of locally homogeneous Lorentzian 3-manifolds // Cent. Eur. J. Math. — 2009. — V. 7(1). — P. 124–139.

| Riemannian metric[3] $(\rho_1, \rho_2, 0)$ | Lorentzian metric[4] $(\rho_1, \rho_2, 0)$ |
|---|---|
|  | Changing $\rho_1 \to -\rho_1$, $\rho_2 \to -\rho_2$, $\rho_3 \to -\rho_3$ |



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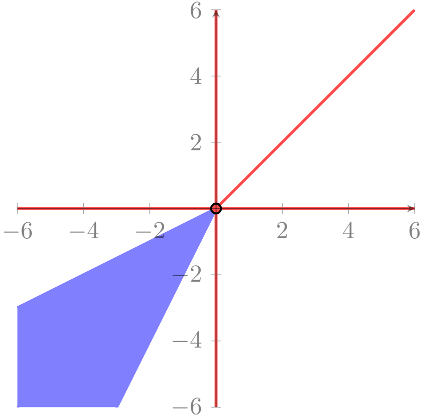
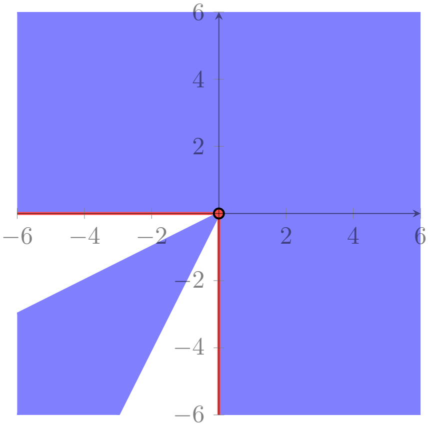
|---|---|
|  |  |





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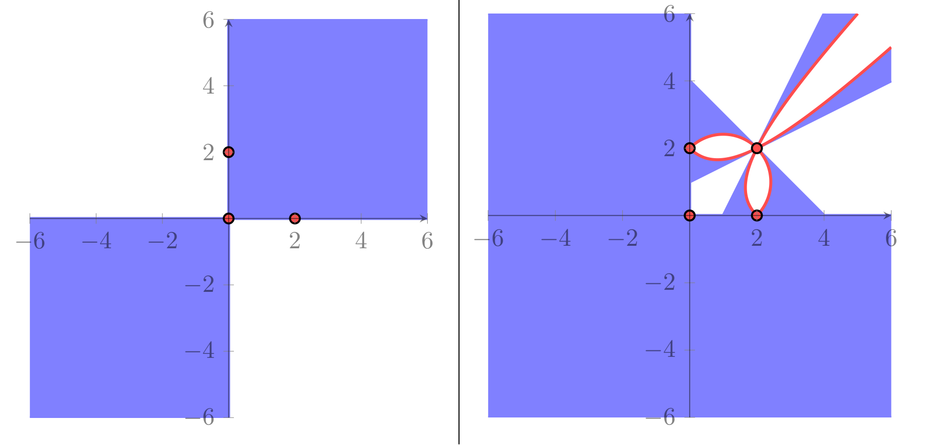
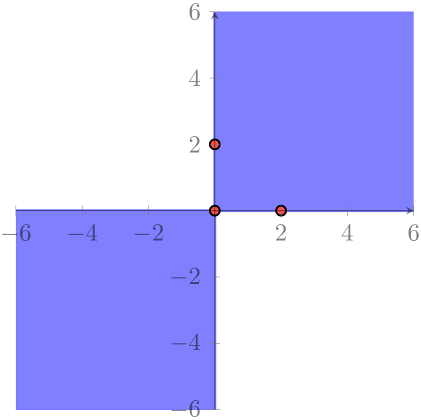
# Possible Ricci eigenvalues. Segre type $\{111\}$

| Riemannian metric[5] $(\rho_1, \rho_2, 2)$ | Lorentzian metric[6] $(\rho_1, \rho_2, 2)$ |
| --- | --- |




[5] Kowalski O., Nikcevic S. On Ricci eigenvalues of locally homogeneous Riemann 3-manifolds // Geom. Dedicata. — 1996. — No. 1. — P. 65–72.

[6] Calvaruso G., Kowalski O. On the Ricci operator of locally homogeneous Lorentzian 3-manifolds // Cent. Eur. J. Math. — 2009. — V. 7(1). — P. 124–139.

| Riemannian metric[5] $(\rho_1, \rho_2, 2)$ | Lorentzian metric[6] $(\rho_1, \rho_2, 2)$ |
| --- | --- |
|  | Changing $\rho_1 \to -\rho_1,\ \rho_2 \to -\rho_2,\ \rho_3 \to -\rho_3$ |



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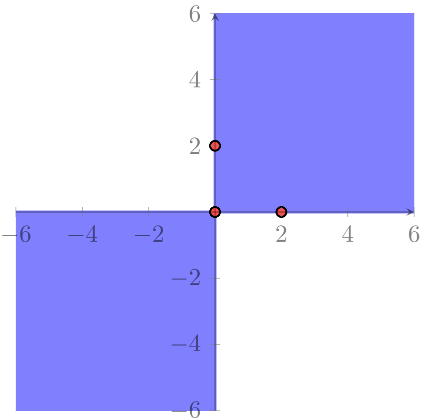
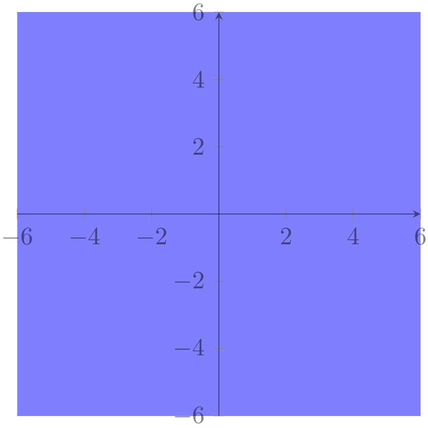
|---|---|
|  |  |

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[6] Calvaruso G., Kowalski O. On the Ricci operator of locally homogeneous Lorentzian 3-manifolds // Cent. Eur. J. Math. — 2009. — V. 7(1). — P. 124–139.

# Conditions of existence the locally homogeneous 3-manifolds

**Theorem**[7]

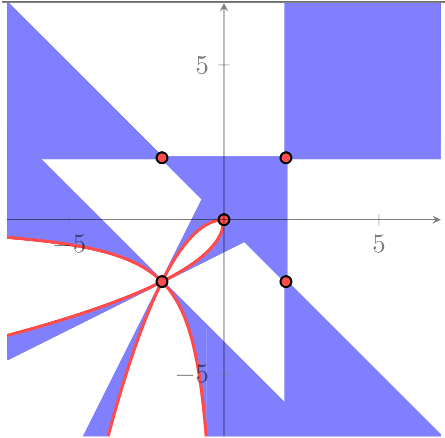
A locally homogeneous Lorentzian 3-manifold $(M, g)$ with non-diagonalizable Ricci operator $\rho$ exists if and only if one the following mutually exclusive statements holds true:

1. $\rho$ is Segre type $\{1z\bar{z}\}$ with the real Ricci eigenvalue $\rho_1 < 0$;
2. $\rho$ is Segre type $\{21\}$ with Ricci eigenvalues $\rho_1 < 0$ or $\rho_1 = \rho_2 = 0$;
3. $\rho$ is Segre type $\{3\}$ with a triple Ricci eigenvalue $\rho_1 < 0$.

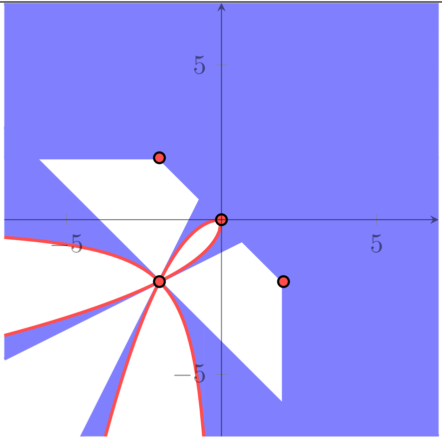
[7] Calvaruso G., Kowalski O. On the Ricci operator of locally homogeneous Lorentzian 3-manifolds // Cent. Eur. J. Math. — 2009. — V. 7(1). — P. 124–139.

# Possible section curvature eigenvalues. Segre type $\{111\}$

Riemannian metric[8] $(k_1, k_2, -2)$

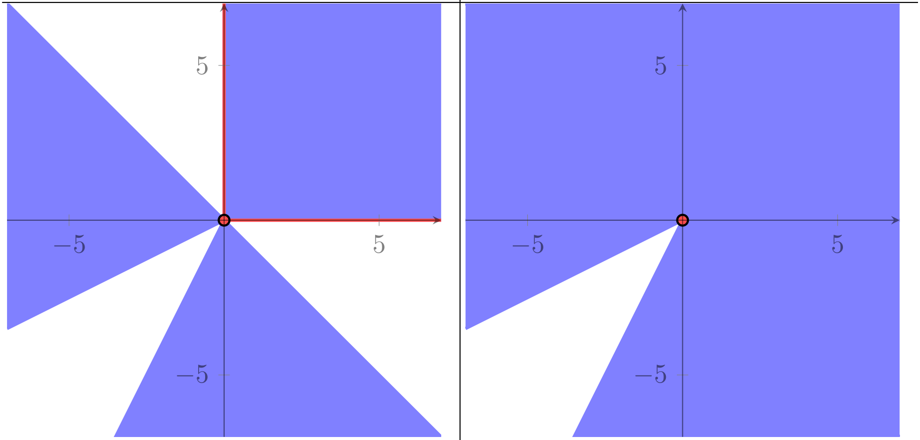



Lorentzian metric $(-k_1, -k_2, 2)$



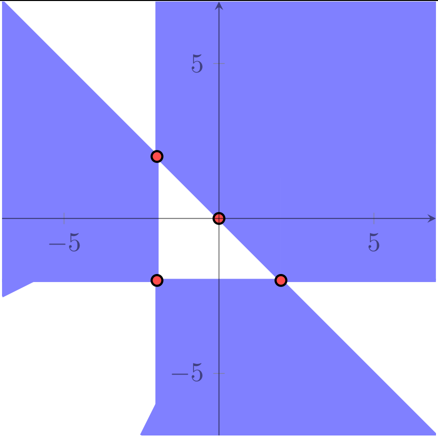
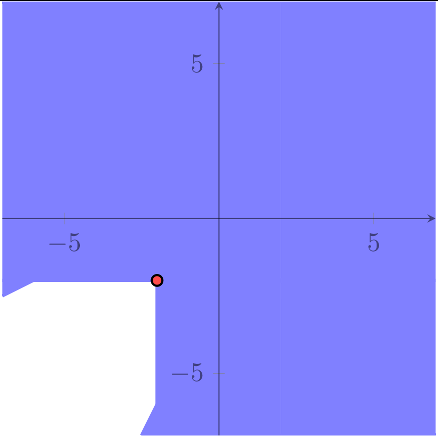

[8] Oskorbin D.N., Rodionov E.D. On the spectrum of the curvature operator of a three-dimensional Lie group with a left-invariant Riemannian metric // Doklady Mathematics. — 2013. — V. 87, No 3. — P. 307–309.

| Riemannian metric[9] $(k_1, k_2, 0)$ | Lorentzian metric $(-k_1, -k_2, 0)$ |
| --- | --- |




[9] Oskorbin D.N., Rodionov E.D. On the spectrum of the curvature operator of a three-dimensional Lie group with a left-invariant Riemannian metric // Doklady Mathematics. — 2013. — V. 87, No 3. — P. 307–309.

# Possible section curvature eigenvalues. Segre type $\{111\}$

| Riemannian metric[10] $(k_1, k_2, 2)$ | Lorentzian metric $(-k_1, -k_2, -2)$ |
| --- | --- |
|  |  |



[10]Oskorbin D.N., Rodionov E.D. On the spectrum of the curvature operator of a three-dimensional Lie group with a left-invariant Riemannian metric // Doklady Mathematics. — 2013. — V. 87, No 3. — P. 307–309.

**Theorem**

A connected, simply connected three-dimensional Lorentzian locally homogeneous manifold $(M, g)$ with non-diagonalizable sectional curvature operator $\mathcal{K}$ exist if and only if $\mathcal{K}$ satisfies one of the following conditions:

1. $\mathcal{K}$ has the Segre type $\{21\}$ and
   1. eigenvalues equal to zero, or
   2. $k_1 = -3k_2 > 0$;
2. $\mathcal{K}$ has the Segre type $\{3\}$ with negative eigenvalue;
3. $\mathcal{K}$ has the Segre type $\{1z\overline{z}\}$ and
   1. complex eigenvalues have negative real part, or
   2. $0 \leqslant \frac{k_2+k_3}{2} < -k_1$.

Thank you for attention!