Schouten-Weyl tensor on locally homogeneous pseudo-Riemannian manifolds

Klepikov P.N.

Altai State University
Barnaul, Russia

Definitions

Let (M, g) be a (pseudo)Riemannian manifold. In standart way, we define Riemann curvature tensor R, Ricci tensor r and scalar curvature s:

$$R(X,Y)Z = [\nabla_Y, \nabla_X]Z + \nabla_{[X,Y]}Z,$$
$$r(X,Y) = \operatorname{tr}(V \to R(X,V)Y),$$
$$s = \operatorname{tr}(\rho).$$

Schouten-Weyl tensor SW (or Cotton tensor) on (pseudo)Riemannian manifold of dimension $n \geqslant 3$ are defined by

$$SW(X,Y,Z) = \frac{1}{n-2}\Bigg(\nabla_Z r(X,Y) - \nabla_Y r(X,Z) - \\ - \frac{1}{2(n-1)}\left(g(X,Y)\nabla_Z s - g(X,Z)\nabla_Y s\right)\Bigg).$$

Weyl-Schouten theorem

Three-dimensional (pseudo)Riemannian manifold is conformally flat manifold iff $SW = 0$.

If the dimension of a (pseudo)Riemannian manifold is greater than three, then

$$\operatorname{div} W = (3-n) SW.$$

If the scalar curvature is a constant (for example when (pseudo)Riemannian manifold is locally homogeneous manifold), then we have

$$SW\left(X,Y,Z\right) = \frac{1}{n-2}\left(\nabla_Z r\left(X,Y\right) - \nabla_Y r\left(X,Z\right)\right).$$

In this case, manifolds with $SW = 0$ coincides with Einstein-like manifolds of type $\mathcal{B}$[1].

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$$SW(X, Y, Z) = 0 \quad \Leftrightarrow \quad \nabla_Z r(X, Y) = \nabla_Y r(X, Z).$$

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Classification problem in four dimensions

Classification of the (pseudo)Riemannian locally homogeneous manifolds with nontrivial isotropy subgroup and zero Schouten-Weyl tensor are obtained by A. Zaeim and A. Haji-Badali[2].
Lie groups with the left-invariant Riemannian metric and zero Schouten-Weyl tensor are classified by O.P. Gladunova, V.V. Slavskii[3], D.S. Voronov and E.D. Rodionov[4].

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[2] Zaeim A., Haji-Badali A. Einstein-like pseudo-Riemannian homogeneous manifolds of dimension four // Mediterr. J. Math. — 2016. — Vol. 13(5). — P. 3455–3468.

[3] Gladunova O.P., Slavskii V.V. Harmonicity of the Weyl tensor of left-invariant Riemannian metrics on four-dimensional unimodular Lie groups // Siberian Advances in math. — 2013. — Vol. 23(1). — P. 32–46.

[4] Voronov D.S., Rodionov E.D. Left-invariant Riemannian metrics on four-dimensional nonunimodular Lie groups with zero-divergence Weyl tensor // Doklady Mathematics. — 2010. — Vol. 81(3). — P. 392–394.

Classification problem in four dimensions

Riemannian case[5]	Pseudo-Riemannian case[6]
Metric Lie algebra $\mathbb{A}_{4,1}$: $[e_2, e_4] = a\, e_1,$ $[e_3, e_4] = b\, e_1 + c\, e_2$ $\Downarrow$ Ricci tensor r $\Downarrow$ $\boxed{\nabla_Z r\,(X, Y) = \nabla_Y r\,(X, Z)}$ system of cubic equations	Let Ricci operator has Segre type $\{112\}$ $\Downarrow$ $\boxed{\nabla_Z r\,(X, Y) = \nabla_Y r\,(X, Z)}$ system of linear equations $+$ Jacobi identity system of quadratic equations
Repeat for all 24 types of metric Lie algebras	Repeat for all 20 posible Segre types

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Ricci soliton

Definition[7]

(Pseudo)Riemannian manifold (M, g) is called a Ricci soliton, if there exists a vector field X and a constant Λ, for which the equation holds:

$$r = \Lambda \cdot g + L_X g.$$

Definition[8]

Metric Lie group (G, g) is called an algebraic Ricci soliton, if there exists a derivation D of Lie algebra $L(G)$ and a constant Λ, for which the equation holds:

$$\rho = \Lambda \cdot \mathrm{Id} + D.$$

Definition

Ricci soliton is called trivial, if it is Einstein manifold or the product of Einstein manifold and (pseudo)Euclidean space.

[7] Hamilton R.S. The Ricci flow on surfaces // Contemporary Mathematics. — 1988. — Vol. 71. — P. 237–262.

[8] Lauret J. Ricci soliton homogeneous nilmanifolds // Math. Ann. — 2001. — Vol. 319(4). — P. 715–733.

Algebraic Ricci soliton with $SW = 0$

Theorem[9]

Let (G, g) be a Lie group with left-invariant (pseudo)Riemannian metric of nontrivial algebraic Ricci soliton and $SW = 0$. Then $\Lambda = 0$ and Ricci operator has Segre type $\{(1 \ldots 1\, 2 \ldots 2)\}$ with a single zero eigenvalue, and amount of "2" in the Segre type is necessarily nonzero.

Riemannian case	Pseudo-Riemannian case
Algebraic Ricci soliton $\Downarrow$ Trivial Ricci soliton	Algebraic Ricci soliton $\Downarrow$ Trivial Ricci soliton or nontrivial

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Conformally flat algebraic Ricci soliton

Theorem[10]

If we fix the signature of the pseudo-Riemannian metric and real number $\alpha \geqslant 0$, then there exists no more than two conformally flat metric Lie algebras (up to isometry), which are nontrivial algebraic Ricci solitons.

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Locally conformally homogeneous manifolds[11]

Locally homogeneous manifolds	Locally conformally homogeneous manifolds
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$$\text{Loc. hom.} \xrightarrow{\forall \text{ conformal transformation}} \text{Loc. conformally hom.}$$

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$n = 3$: $\|SW\|^2 = SW_{ijk}SW^{ijk} \neq 0$;
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Isotropic Schouten-Weyl tensor

Definition

Schouten-Weyl tensor SW is called isotropic if $SW \neq 0$ and $\|SW\|^2 = 0$.

Riemannian case	Pseudo-Riemannian case[12]
$\|SW\|^2 = 0 \Rightarrow SW = 0$	$\|SW\|^2 = 0 \not\Rightarrow SW = 0$
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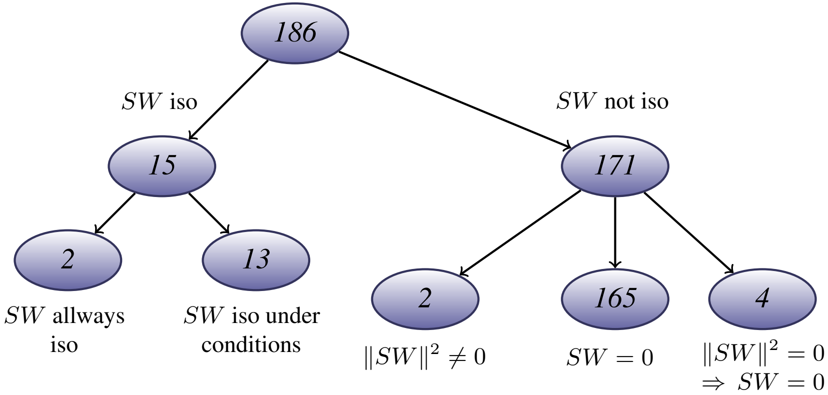
Isotropic Schouten-Weyl tensor in 4D[13,14]



[13] Komrakov B.B. Einstein–Maxwell equation on four-dimensional homogeneous spaces // Lobachevskii J. Math. — 2001. — V. 8. — P. 33–165.

[14] Zaeim A., Haji-Badali A. Einstein-like pseudo-Riemannian homogeneous manifolds of dimension four // Mediterr. J. Math. — 2016. — Vol. 13(5). — P. 3455–3468.

Thank you for attention!