Realisation of subgroups of G_2^* as holonomy groups

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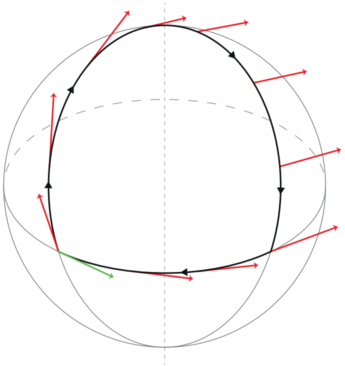
Holonomy of a manifold

- Parallel transport along loops
- Global property
- Property of the connection
- Semi-Riemannian manifold $\implies$ Levi-Civita-connection



Holonomy of semi-Riemannian manifolds

- Locally symmetric manifolds (Cartan, 1926)
- Irreducible semi-Riemannian manifolds (Berger, 1954)

n	(r, s)	holonomy group(s)
$2m \geq 4$	$(2p, 2q)$	$U(p, q)$ or $SU(p, q)$
$2m \geq 4$	(p, p)	$SO(r, \mathbb{C})$
$2m \geq 8$	$(4p, 4q)$	$Sp(p, q)$ or $Sp(p, q) \cdot Sp(1)$
$2m \geq 8$	$(2p, 2p)$	$Sp(p, \mathbb{R}) \cdot SL(2, \mathbb{R})$
$2m \geq 16$	$(4p, 4p)$	$Sp(p, \mathbb{C}) \cdot SL(2, \mathbb{C})$
7	$(4, 3)$	G_2^*
14	$(7, 7)$	G_2^c
8	$(4, 4)$	$Spin(4, 3)$
16	$(8, 8)$	$Spin(7, \mathbb{C})$

- Indecomposable pseudo-Riemannian manifolds
  $\Longrightarrow$ only special geometries

Holonomy and curvature

Ambrose-Singer holonomy theorem

Let M be a analytic, pseudo-Riemannian manifold with holonomy H. Then, the Lie algebra $\mathfrak{h}$ of H in $x \in M$ is spanned by the curvature endomorphisms $R_x(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$.

- The Lie algebra $\mathfrak{h}$ of H is called *holonomy algebra*
- For a basis $b_1, \ldots, b_n$ of $T_x M$ denote

$$R_{ij} = R(b_i, b_j) \subset \mathfrak{h} \quad i, j = 1, \ldots, n$$

About the group G_2^*

- Lie theory: simple, complex Lie groups
  - 4 series of classical groups $SL(n), SO(2n), SO(2n+1), Sp(2n)$
  - 5 exceptional groups G_2, F_4, E_6, E_7, E_8

- G_2 is the automorphism group of $\mathbb{O}$

- $\mathfrak{g}_2$: complex, 14-dim., with 2 real forms
  - compact form $\mathfrak{g}_2^c \Leftrightarrow G_2^c$
  - split-form $\mathfrak{g}_2^* \Leftrightarrow G_2^*$

Characterisation of G_2^*

Let $b_1, \ldots, b_7$ be a basis of a real vector space with $\dim V = 7$.

- Stabiliser of a certain 3-form ω_0 (Engel, 1900) with

$$\omega_0 = \sqrt{2}(b^{167} + b^{235}) - b^4 \wedge (b^{15} - b^{26} - b^{37})$$

- ω_0 induces a metric with signature (4,3) on M

$$g = 2(b^1 \cdot b^5 + b^2 \cdot b^6 + b^3 \cdot b^7) - (b^4)^2$$

- $G_2^* \subset SO(4,3)$ acts transitively on isotropic lines in $\mathbb{R}^{4,3}$

Classification of holonomy subgroups of G_2^*

Let M be a connected, simply-connected, real-analytic manifold with dimension $n = 7$. Let the holonomy H of M be contained in G_2^*. The holonomy representation is indecomposable but not irreducible.

- Classification via holonomy algebra $\mathfrak{h}$ of H
- $H \subset G_2^* \implies \omega_0 \implies g$
- Algebraic conditions on $\mathfrak{h}$:
  - Indecomposability
  - Berger's criterion
  - *socle*
- Distinction by the dimension of the socle: Type I, Type II, **Type III**

Typ III holonomy algebras

$$\begin{pmatrix} \operatorname{tr} A & 0 & 0 & \sqrt{2}v & 0 & -y_1 & -y_2 \\ 0 & a_1 & a_2 & 0 & y_1 & 0 & v \\ 0 & a_3 & a_4 & 0 & y_2 & -v & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}v & 0 & 0 \\ 0 & 0 & 0 & 0 & -\operatorname{tr} A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_1 & -a_3 \\ 0 & 0 & 0 & 0 & 0 & -a_2 & -a_4 \end{pmatrix} \in \mathfrak{h} \subset \mathfrak{g}_2^*$$

Classification theorem for Type III algebras

If the holonomy algebra $\mathfrak{h}$ is of Type III, then a basis of V exists such that

a.) $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$ with $\mathfrak{a} \in \{\mathfrak{gl}(2), \mathfrak{sl}(2), \mathfrak{co}(2), \mathfrak{d}\}$

b.) $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}(1,k)$ with $\mathfrak{a} \in \{0, \mathbb{R} \cdot \operatorname{diag}(1,0)\}$ and $k = 1, 2$.

Realisation of holonomy subgroups

Example: $\mathbb{R} \cdot \mathrm{diag}(1,0) \ltimes \mathfrak{m}(1,2)$

Let M be a manifold with holonomy $H \subset G_2^*$ of Type III and $V = T_0M$. Assume $\mathfrak{h} \subset \mathbb{R} \cdot \mathrm{diag}(1,0) \ltimes \mathfrak{m}(1,2)$:

1. $H \subset G_2^* \implies \exists \omega \in \Lambda^3 V^* : \omega \simeq \omega_0$
   $\implies g = 2(b^1 \cdot b^5 + b^2 \cdot b^6 + b^3 \cdot b^7) - (b^4)^2$
2. $g \implies \nabla^{LC}$ Levi-Civita-connection $\implies$ local connection-form θ
3. Cartan's first structure eq.: $\mathrm{d}b = -\theta \wedge b$

Structure eq. and exterior differential systems

$$\begin{pmatrix} db^1 \\ db^2 \\ db^3 \\ db^4 \\ db^5 \\ db^6 \\ db^7 \end{pmatrix} = - \begin{pmatrix} \mathbf{x} & 0 & 0 & \sqrt{2}\mathbf{v} & 0 & -\mathbf{y}_1 & -\mathbf{y}_2 \\ 0 & \mathbf{x} & 0 & 0 & \mathbf{y}_1 & 0 & \mathbf{v} \\ 0 & 0 & 0 & 0 & \mathbf{y}_2 & -\mathbf{v} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}\mathbf{v} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} b^1 \\ b^2 \\ b^3 \\ b^4 \\ b^5 \\ b^6 \\ b^7 \end{pmatrix}$$

Methods of exterior differential systems (Frobenius' theorem) lead to:

$$\begin{aligned} b^1 &= dx_1 + r_5(x_1, x_2, x_4, x_5, x_6, x_7) \cdot dx_5 + r_6(x_2, x_5, x_6, x_7) \cdot dx_6 \\ b^6 &= dx_6 + v(x_5, x_6) \cdot dx_5 \\ b^i &= dx_i \qquad\qquad i = 2, 3, 4, 5, 7 \end{aligned}$$

for $x_1, \ldots, x_7$ local coordinates on M.

4. The components of the connection-form θ

$$\theta_{ij} = b^i(\nabla b_j)$$

generate a PDES for v, r_5, r_6:

$$\begin{aligned} 2(r_5)_{x_1} &= -(v)_{x_6} + (r_6)_{x_2} \\ \sqrt{2}(r_5)_{x_4} &= (r_6)_{x_7} \\ -(v)_{x_6} &= (r_6)_{x_2} \\ (r_5)_{x_2} &= v \cdot (r_6)_{x_2} \end{aligned}$$

Solution restricts the functions shape:

$$\begin{aligned} r_5 &= \frac{x_4}{\sqrt{2}} \cdot (F_6)_{x_7} - x_2 \cdot v \cdot (v)_{x_6} - x_1 \cdot (v)_{x_6} + F_5(x_5, x_6, x_7) \\ r_6 &= -x_2 \cdot (v)_{x_6} + F_6(x_5, x_6, x_7) \end{aligned}$$

Local metric with holonomy $\mathbb{R} \cdot \mathrm{diag}(1,0) \ltimes \mathfrak{m}(1,2)$

Choose v, F_5, F_6 as follows:

$$v = \frac{1}{2}x_6^2 \qquad F_6 = -x_7^2 \qquad F_5 = 0$$

and compute the curvature endomorphisms

$$R_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_{56} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_{57} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_{67} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thank you for your attention!