Factorisation of the conformal higher-spin operators

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Structure of the talk

- Ambient space. Construction of GJMS operators. Factorisation in terms of tractors.
- Conformal higher spin operators: manifestly factorised form and gauge transformations.

Ambient space[Dirac, 1935; Klein, 19th century]

Definition:

- $n + 2$ dim Minkowski-like space $\mathbb{R}^{n,2}$
- Equipped with quadratic form $Q = \sum_{\alpha=1}^{n} (X_\alpha)^2 - (X_0)^2 - (X_{n+1})^2$
- Metric $\eta_{AB} = diag(-, +, +..., +, -)$

Cone is a zero locus $X^2 = 0$. η_{AB} defines a conformal structure on the cone.

Why $SO(n,2)$?

- Pullbacks of metric η_{AB} on different sections of the cone are conformally related.
- consider scalar field $\phi(x)$ on particular section of the cone. Extend along the rays via $(X\frac{\partial}{\partial X} - w)\phi = 0$.
- $SO(n,2)$ preserves the cone, maps rays into another rays $\Rightarrow$ one recover conformal transformations.

Tractors in ambient picture

Choose a section of the cone Σ. Let Φ be a section from Σ to $T\mathbb{R}^{n,2} \otimes T\mathbb{R}^{n,2} \otimes ... \otimes T\mathbb{R}^{n,2}$. Extend to the entire ambient space via

$$(X\frac{\partial}{\partial X} - w)\Phi(X) = 0 \tag{1a}$$

$$\Phi(X) \sim \Phi(X) + X^2\chi(X) \tag{1b}$$

Denote the space of all solutions 1a,b $\mathcal{E}^{\bullet}[w]$.
•**Proposition**[Gover, 02]:
$\mathcal{E}^{\bullet}[w]$ is a space of tractor tensors with weight w.
Well-defined on the equivalence classes 1b Thomas D-operator $D^A : \mathcal{E}^{B..C}[w] \mapsto \mathcal{E}^{AB..C}[w-1]$.
[Bailey, Eastwood, Gover, 1994]:

$$D_A = 2(X\frac{\partial}{\partial X} + \frac{n}{2})\frac{\partial}{\partial X^A} - X_A\Delta \tag{2}$$

GJMS operators in ambient space

Definition of conformally invariant operator

A linear differential operator P^g acting on a function or tensor/spinor field is said to be a conformally invariant operator if for all positive functions Ω:
$P^{\hat{g}} \circ \Omega^{w_1} = \Omega^{w_2} \circ P^g$
where $\hat{g} = \Omega^2 g$. In ambient picture, it must be equivariant to $SO(n,2)$ transformations.

Let us consider $sl(2)$ relations between next operators:
$h = X\frac{\partial}{\partial X} + \frac{1}{2}(n+2)$, $x = -\frac{1}{4}X^2$, $y = \frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X}$. The commutation relations are $[x,y] = h$, $[h,x] = 2x$, $[h,y] = -2y$.

GJMS operators in ambient space

$$(X\frac{\partial}{\partial X} - w)\Phi(X) = 0 \tag{3a}$$

$$\Phi(X) \sim \Phi(X) + X^2\chi(X) \tag{3b}$$

Consider operator $\Delta^k = \frac{\partial}{\partial X}\frac{\partial}{\partial X}$ acting on $\mathcal{E}^\bullet[w]$. It is well-defined on the equivalence classes and thus descends to conformal invariant operator on Σ. We'll denote it P^{2k}

Proof:

$$\Delta^k(X^2\chi) = X^2\Delta^k\chi + 4k\Delta^{k-1}(w_\chi + \frac{n}{2} - k + 2)\chi = X^2\Delta^k\chi \tag{4}$$

GJMS operators in ambient space

$$(X\frac{\partial}{\partial X} - w)\Phi(X) = 0 \tag{5a}$$

$$\Phi(X) \sim \Phi(X) + X^2\chi(X) \tag{5b}$$

One can demand a harmonic condition on the Φ:

$$\Delta\Phi = 0 \tag{6}$$

Extension is unique up to $\Phi \sim \Phi + (X^2)^k\chi$.($sl(2)$ relations) So, $(X^2)^{1-k}\Delta\Phi|_\Sigma$ defines a conformally invariant operator. The same $sl(2)$ relations imply that it is P^{2k} [Graham,..., 1992]

Tractor formulas for GJMS operators

- While acting on $\Phi \in \mathcal{E}^\bullet[k - \frac{n}{2}]$:
  $D_{A_1} \dots D_{A_s} \Phi = (-1)^k X_{A_1} \dots X_{A_s} P^{2k} \Phi +$ Weyl curvature terms [Gover, 02]
- Choosing an Einstein section of the cone one can introduce scale tractor $I := \nabla^T_\mu I = 0$

$$I^{A_1} \dots I^{A_s} D_{A_1} \dots D_{A_s} \Phi = I^{A_1} D_{A_1} \dots I^{A_s} D_{A_s} \Phi = P^{2k} \Phi \tag{7}$$

Since $I^A X_A = 1$

- $P^{2k} \Phi = (ID)^k \Phi, \Phi \in \mathcal{E}^\bullet[k - \frac{n}{2}]$ [Gover, 06]

Who is factorised? Not Δ^k, but $(ID)^k$.

Higher spin fields

Let us consider a conformal field $\phi(x,p) = p^{a_1}...p^{a_s}\phi(x)_{a_1...a_s}$ tangential to the Σ. What are the conformally invariant operators for such fields?

- Fradkin, Tseytlin 1985
- Segal, 2002

Manifestly $O(n,2)$ invariant description

- Bekaert, Grigoriev 2013

Notion of factorisation

- Metsaev

Manifestly factorised expression for FT fields

- Tseytlin 2013 (gauge-fixed)
- Nutma, Taronna 2015 (guessed expression for a particular weight $w = s - 2$)

CHS fields in the ambient space

- A way to encode CHS fields in tractor tensor field

$$
\begin{gathered}
(X\frac{\partial}{\partial X} - w)\Phi = 0, \;\; X\frac{\partial}{\partial P}\Phi = 0 \\
\Phi \sim \Phi + X^2\chi, \;\; D\frac{\partial}{\partial P}\Phi = 0 \\
\frac{\partial}{\partial P}\frac{\partial}{\partial P}\Phi = 0, \;\; P\frac{\partial}{\partial P}\Phi = s\Phi
\end{gathered} \tag{8}
$$

Instead of $I\mathcal{D}$ we have $B := I\mathcal{D} - \frac{1}{s-1-w}P^A\mathcal{D}_A I^B \frac{\partial}{\partial P^B}$ - well-defined on all the constraints and equivalence relations.

CHS operator, gauge transformations

- CHS operator on constant curvature sections of the cone is just B^k acting on $\Phi(X,P)$ of weight $w = k - \frac{n}{2}$. Use lift T to write in metric notation:
  $B^k\Phi(X,P) = T^{-1}BTT^{-1}...BT^{-1}\Phi(X,P)$.

In metric notations:

$$\prod_{i=0}^{\frac{n-4}{2}+s-t+1} \{\Box^0 + \frac{2J}{n}(-s+(n+w-i-1)(w-i)) - \\ -\frac{n+2s-4}{(s-1-w+i)(n+s+w-i-2)}(p\nabla^0)(\frac{\partial}{\partial p}\nabla^0) + \\ +\frac{1}{(n+s+w-i-2)(s-1-w+i)}p^2(\frac{\partial}{\partial p}\nabla^0)^2\}\Phi(x,p) \tag{9}$$

Gauge relations:

$$B(P\mathcal{D})^t\lambda \sim (P\mathcal{D})^{t+1}I\frac{\partial}{\partial P}\lambda \tag{10}$$