HOMOGENEOUS GEOMETRIC STRUCTURES

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Lecture 5: Cartan geometries

$(G \to G/H, \omega_G)$ ... Klein geometry

$(\mathcal{G} \to M, \omega)$ ... Cartan geometry of type (G,H) if $\mathcal{G}$ is H-bundle over M, $\omega : T\mathcal{G} \to \mathfrak{g}$, iso on $T_u\mathcal{G}$

$\omega(\zeta_Z) = Z \quad \forall Z \in \mathfrak{h}$, $\omega(wh) = Ad_h^{-1}\omega(w)$.

Curvature $\mathcal{K}(X,Y) = d\omega(\omega^{-1}(X), \omega^{-1}(Y)) + [X,Y]$ ... $\mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$

= failure to satisfy Maurer Cartan equation

$\omega^{-1}(X)$ ... constant vector fields = generalization of left invariant vector field $\Rightarrow$ defines normal coordinates $\mathfrak{g} \to \mathcal{G}$ as generalization of $\exp: \mathfrak{g} \to G$

Examples

1) $(\mathbb{R}^n \rtimes GL(n,\mathbb{R}), GL(n,\mathbb{R}))$ ... affine geometries
$\mathcal{G} = P^1M$, $\omega = \theta + \gamma$ ... defines a linear connection ∇^γ

2) 1-st order G_0-structures $\overset{1:1}{\Longleftrightarrow}$ Cartan geometries of type $(\mathbb{R}^n \rtimes G_0, G_0)$ such that $\mathcal{K} = T + R$ with T in complement $\mathcal{H} \subset \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ of $Im(\partial)$

3) $(PGL(n+1), GL(n) \rtimes \mathbb{R}^{n*})$ ... projective geometries
+ $\mathcal{K}$ has values in $\mathfrak{h}$ = torsion free
i.e. $[\nabla]$ ... share geodesics up to parametrization

4) $(PSO(p+1,q+1), CSO(p,q) \rtimes \mathbb{R}^{n*})$ ... conformal geometries
+ torsion free
$[g]$ .. $\tilde{g} = e^f g$, $f: M \to \mathbb{R}$ smooth
... allows to measure angle but not distance

5) $(PSU(p+1,q+1), CSU(p,q) \rtimes \mathbb{C}^{n*} \rtimes \mathbb{R})$ ... $M \subset \mathbb{C}^{n+1}$ hypersurface
+ $\partial^*\mathcal{K} = 0$ for certain linear operator
$Re([\xi, i\eta])$: $(TM \cap iTM) \otimes (TM \cap iTM) \to \mathbb{R}$ ... signature (p,q)

Homogeneous Cartan geometries of type (G,P)

= pair of maps $\alpha : \underline{k} \to \underline{g}$ (linear), $i : H \to P$ (homomorphism)

$\alpha(\underline{k}/\underline{h}) = \underline{g}/\underline{p}$, $\alpha(Ad(h)X) = Ad(i(h))\alpha(X)$

$\alpha|_{\underline{h}} = di$

$\Rightarrow$ $\mathcal{G} = K \times_{i(H)} P$ and $\omega_\alpha|_{TK} = \alpha \circ \omega_K$

$\downarrow$ $\qquad \omega_\alpha|_{TP} = \omega_P$

K/H $\qquad \omega_\alpha(k,p) = Ad_p^{-1}(\omega_\alpha(k,e))$

Natural bundles: $\rho : P \to GL(\mathbb{V})$, $\rho \circ i : H \to GL(\mathbb{V})$

$s : \mathcal{G} \to \mathbb{V}$ equivariant $\Leftrightarrow$ $s : K \to \mathbb{V}$ $s(kh) = \rho\circ i(h^{-1})s(k)$

K invariant $\Leftrightarrow$ $s(k) = s(e)$ $\forall k$ – uniquely determined by $\mathbb{V}^H = \{v, Hv = v\}$

Natural differentiation = Fundamental derivative D $s \in \Omega^1(K, \mathbb{V})$

$D_X s = \omega^{-1}(X)(s)$, K invariant $\Leftrightarrow$ $Ds = 0$

$D_Y s = -d\rho \circ \alpha(Y)(s)$, $Y \in \underline{k}$

K-invariant connection $\nabla^{\Phi} = D + \Phi$, $\Phi \in (\underline{k}^* \otimes End(\mathbb{V}))^H$

$\Phi(Y) = d\rho \circ \alpha(Y)$

Tractor bundle $\lambda : \underline{g} \to End(\mathbb{V})$ representation, $\lambda|_{\underline{p}} = d\rho$, P-equivariant

$\Rightarrow$ $\Phi = \lambda \circ \alpha$ is K-invariant connection

Parallel sections $\nabla^{\Phi} s = 0$, ... infinitesimal holonomy

determines locally by value at one point (by real analyticity)

$S^0 = \{ v \in \mathbb{V}, R^{\Phi}(\omega) v = 0, \phi(X)\phi(Y)(v) - \phi(Y)\phi(X)(v)$

$- \phi(X,Y) = 0 \ \forall X, Y \in \underline{k} \}$

$S^1 = \{ v \in S^0, \phi(X) v \in S^0, \forall X \in \underline{k} \}$ for some i

$S^i = \{ v \in S^{i-1}, \phi(X) v \in S^{i-1}, \forall X \in \underline{k} \}$ .. $S^i = S^{i+1} = \dots S^\infty$

$\Rightarrow$ S^∞ is representation of $\underline{k}$, trivial represent = K-invariant solution

Locally $v \in S^\infty$ $s(\exp(X)) = \exp^{-\Phi(X)}(v)$ in exponential coordinates

Globally problems to extend representation of $\underline{k}$ to representation of K.

Example

$K = \begin{pmatrix} 1 & 0 & 0 \\ y_1 & 1 & 0 \\ y_3 & y_2 & 1 \end{pmatrix}$ $H = id$

$$X_1 := \partial_{y_1} + y_2\partial_{y_3}, \quad X_2 := \partial_{y_2}, \quad X_3 := \partial_{y_3},$$ – left invariant v.f.

$$\theta_1 := dy_1, \quad \theta_2 := dy_2, \quad \theta_3 := -y_2 dy_1 + dy_3,$$ – Maurer Cartan form $\to \mathfrak{k}$

Projective structure $K \times \begin{pmatrix} a & z \\ 8 & A \end{pmatrix} = P_1$ $(PGL(4,\mathbb{R}), P_1)$

$\sigma \left(\downarrow\uparrow \right)$

K

$\sigma^* \omega$ = 4x4 matrix of 1-forms = Cartan connection

K - homogeneous $\sigma^* \omega = \alpha \circ \omega_K = \begin{pmatrix} a^k \theta_k & P^k_j \theta_k \\ \theta_1 & \\ \theta_2 & A^{jk}_i \theta_k \\ \theta_3 & \end{pmatrix}$ – constant

$$\alpha(x_1X_1 + x_2X_2 + x_3X_3) := \begin{bmatrix} a^k x_k & P^k_j x_k \\ x_i & A^{jk}_i x_k \end{bmatrix}$$

$a = 0$, A^{jk}_i ... connection – has to be torsion free

P^k_{ij} ... Rho tensor – determined by $\sigma^* \mathcal{K} = 0$

For example

$$\tau^*\omega = \begin{bmatrix} 0 & \theta_1 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 \\ \theta_2 & -\theta_2 & -\theta_1 & 0 \\ \theta_3 & \theta_3 & -\theta_1 & \theta_1 \end{bmatrix}, \quad \tau^*\kappa = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4\theta_1\wedge\theta_2 & 0 & 0 \end{bmatrix},$$

where $\theta_1 \wedge \theta_2(X,Y) = \frac{1}{2}(\theta_1(X)\theta_2(Y) - \theta_1(Y)\theta_2(X))$.

Question, is there Levi-Civita connection in the projective class?

$\mathcal{V} = S^2\mathbb{R}^4$ ... tractor bundle ... $\nabla\Phi$ ...

nondegenerate parallel section $\Leftrightarrow$ Einstein metrics with Levi-Civita connection in the projective class

$\Rightarrow$

$$\mathcal{S}^0 = \{ \left[\begin{array}{c|ccc} w_1 & 0 & w_4 & w_7 \\ \hline * & 0 & 0 & 0 \\ * & * & w_6 & w_9 \\ * & * & * & w_{10} \end{array}\right] \} \qquad \mathcal{S}^1 = \mathcal{S}^\infty = \{ \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline * & 0 & 0 & 0 \\ * & * & w_6 & w_9 \\ * & * & * & w_{10} \end{array}\right] \}.$$

$\Rightarrow$ solutions are degenerate