# HOMOGENEOUS GEOMETRIC STRUCTURES

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Lecture 4: Symmetric spaces

$\mathbb{R}^n$, central symmetry $x \mapsto -x$, $0$ fixed point

$M$, $S_x$ ... symmetry at $x$ = diffeomorphism

$S_x^2 = id$, $S_x x = x$, $x$ ... isolated fixed point

$(M,S)$ symmetric space $S: M \times M \to M$ smooth such that

1) $S_x$ ... symmetry at $x$ ... $S$ = system of symmetries

2) $S_x S_y z = S_{S_x y} S_x z$ ... $S_x$ is automorphism of $S$

Example: $\mathbb{R}^n$: $S_x(y) = 2x - y$

$G(S)$ ... group generated by symmetries

Theorem: $M$ is homogeneous space $G(S)/H$ and there is involution $\sigma: G(S) \to G(S)$, i.e. $\sigma^2 = id$, $\sigma(hg) = \sigma(h)\sigma(g)$ such that

1) $H$ is open subgroup of fixed point set of $\sigma$

2) $\sigma(g) = s g s^{-1}$ for $s = S_{eH} \in H$.

3) $S_{gH} kH = g s g^{-1} kH$

Proof: $\frac{1}{2} \frac{d}{dt}\Big|_{t=0} S_{c(t)} S_x x = X$ for $c(0) = x, \dot{c}(0) = X$

$\Rightarrow G(S)$ acts transitively on $M$, $H$ stabilizer of $x$

$R_X(y) = \frac{1}{2} \frac{d}{dt}\Big|_{t=0} S_{c(t)} S_x y$ ... vector fields on $M$

their brackets generate finite dimensional algebra $\mathfrak{g}$

$\Rightarrow$ this is Lie algebra of $G(S)$ + results of Palais $\Rightarrow$

$\Rightarrow$ Lie group of transformations of $M \Rightarrow M = G(S)/H$ $\square$

$(G,H,s)$ ... homogeneous symmetric space if 1),2),3) hold

$\mathfrak{m} \subset \mathfrak{g}$ spanned by $R_X$ ... complement to $\mathfrak{h}$ in $\mathfrak{g}$

The following is equivalent: 1) $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{m},\mathfrak{h}] \subset \mathfrak{m}$, $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$

2) $Ad_s|_{\mathfrak{m}} = -id$, $Ad_s|_{\mathfrak{h}} = id$

$G \to G/H$, $TG = G \times (\underline{m} + \underline{h})$

$\gamma(g)(0_m, \ell_Z) = Z$ $\quad Z \in h$ ... connection

Torsion of $\gamma$ $\quad T = 0$

curvature of $\gamma$ $\quad R = -[[R_X, R_Y], R_Z]$

$\gamma$ induces connection on $TG/H$ by $\nabla_X Y = [R_X, Y]$, $\nabla R = 0$

Prime symmetric space $(G, H, s) \Leftrightarrow G = G(S) \Rightarrow \underline{h} = [\underline{m}, \underline{m}]$

Affine symmetric space $(M, \nabla)$, $T = 0$, $\nabla R = 0$

$\nabla_{\dot c(t)} \dot c(t) = 0$ ... geodesics, need to be complete for $S_x$ globally defined as geodesic reflexion $c(t) \mapsto c(-t)$, $c(0) = x$ geodesic

$c(t) = \exp(X) kH$, $c(0) = kH$, $\dot c(0) = R_X(eH)$

Examples: Involutions of matrices and simple symmetric spaces

Inner: $s = \begin{pmatrix} -id & 0 \\ 0 & id \end{pmatrix}$ in correct basis (there are more possibilities)

$GL(n, \mathbb{R}) / GL(p, \mathbb{R}) \times GL(q, \mathbb{R})$ or $\mathbb{C}, \mathbb{H}, Sp, SO^*$

$SO(r, s) / SO(k, l) \times SO(r-k, s-l)$ or $U$ or $Sp$

$SU(n, n)/GL(n, \mathbb{C})$, $SL(n, \mathbb{H})/GL(n, \mathbb{C})$, $SO(2n, \mathbb{C})/GL(n, \mathbb{C})$, $SO(n, n)/GL(n, \mathbb{R})$

$SO^*(4n)/GL(n, \mathbb{H})$, $Sp(2n, \mathbb{C})/GL(n, \mathbb{C})$, $Sp(n, n)/GL(n, \mathbb{H})$, $Sp(2n, \mathbb{R})/GL(n, \mathbb{R})$

outer: $\sigma(A) = (A^T)^{-1}$: $\quad \sigma(AB) = (B^T A^T)^{-1} = (A^T)^{-1} (B^T)^{-1} = \sigma(A) \cdot \sigma(B)$

or analogies involving signature

$GL(n, \mathbb{C})/SO(n, \mathbb{C})$, $GL(n, \mathbb{R})/SO(p, q)$

$SU(p, q)/SO(p, q)$, $SO^*(2n)/SO(n, \mathbb{C})$, $SO(n, n)/SO(n, \mathbb{C})$

$\sigma(A) = (\bar{A}^T)^{-1}$ or analogies involving signature

$SO(2p, 2q)/U(p, q)$, $SO^*(2n)/U(p, q)$

$Sp(p, q)/U(p, q)$, $Sp(2n, \mathbb{R})/U(p, q)$

$SU(2p, 2q)/Sp(p, q)$, $SL(n, \mathbb{H})/Sp(p, q)$

$\sigma$: conjugation defining real form

$GL(n, \mathbb{C})/GL(n, \mathbb{R})$ or $\mathbb{H}$ $\quad GL(n, \mathbb{C})/U(p, q)$

$SO(n, \mathbb{C})/SO(p, q)$ or $SO^*(2n)$

$Sp(2n, \mathbb{C})/Sp(p, q)$ or $Sp(2n, \mathbb{R})$

There are further more complicated possibilities for $\sigma$

$G_0$-structures on symmetric spaces $(G, H, \sigma)$, $G = G(\mathfrak{s})$

$= \alpha : \underline{m} \to \mathbb{R}^n$ s.t. $i(H) \subset G_0$

$= \rho(h)\, \alpha^* N = \alpha^* N$ for the normal form $N$ $\forall h \in H$

$= \rho(H)$-invariant elements of $\mathbb{V}$ ($G_0$-conjugated to $N$)

If $H$ is connected, then $d\rho(h)(w) = 0 \iff$ above.

5 types of symmetric spaces with $\mathfrak{g}$ simple:

| | | $\mathrm{End}(\underline{m})^{\mathfrak{h}}$ | $\mathrm{Sym}(\underline{m})^{\mathfrak{h}}$ | $\mathrm{Asym}(\underline{m})^{\mathfrak{h}}$ |
|---|---|---|---|---|
| a) | generic | $=\mathbb{R}$ | $=\mathbb{R}$ | $=0$ |
| b) | straight complex | $\mathbb{C}$ | $\mathbb{C}$ | $0$ |
| c) | twisted complex | $\mathbb{C}$ | $\mathbb{R}$ | $\mathbb{R}$ |
| d) | twisted paracomplex | $\mathbb{R} \times \mathbb{R}$ | $\mathbb{R}$ | $\mathbb{R}$ |
| e) | all structures exist | $\mathbb{C} \times \mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ |

$\Rightarrow$ Every simple symmetric space admits $SO(p,q)$-structure

- signature = failure of $H$ to be compact

$\Rightarrow \mathrm{End}(\underline{m})^{\mathfrak{h}}$ contains $\mathbb{C} \Rightarrow$ there is $GL(n,\mathbb{C})$-structure

straight $\Rightarrow SO(n,\mathbb{C})$ structure

twisted $\Rightarrow U(p,q)$ structure (Kähler) $\Rightarrow$ symplectic

$\Rightarrow$ paracomplex $J^2 = id \Rightarrow$ (para-Kähler) $\Rightarrow$ symplectic

<u>How to recognise this</u>: 1) $H$ does not have center (subgroup commuting with every $g$)

$\Rightarrow$ a), b)

1.1) $\underline{m}$ is real representation of $H \Rightarrow$ a) Ex $GL(n,\mathbb{R})/SO(p,q)$

1.2) $\underline{m}$ is complex representation of $H \Rightarrow$ b) Ex $Sp(n,\mathbb{C})/Sp(n)$

2) $H$ has compact center $\Rightarrow$ c) Ex $SO(n+2)/SO(2) \times SO(n)$

3) $H$ has non-compact center $\Rightarrow$ d/e)

3.1) center is dim 1 $\Rightarrow$ d) $SO(1,n+1)/SO(1,1) \times SO(n)$

3.2) — " — 2 $\Rightarrow$ e) $SO(n+2,\mathbb{C})/SO(2,\mathbb{C}) \times SO(n)$

There are no $GL(n,\mathbb{H})$ structures on symmetric space

$\Rightarrow$ no $SO^*(2n)$, $Sp(p,q)$-structures

There are $Sp(1)Sp(p,q)$-structures = Wolf spaces = there is $Sp(1)$-factor in $H$

$Sp(1)SO^*(2n)$-structure on $SO^*(2n+2)/SO^*(2) \times SO^*(2n)$