# HOMOGENEOUS GEOMETRIC STRUCTURES

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## LECTURE 3: G-structures

First order frame bundle $P^1M$ of smooth manifold $M$ $\quad P^1M \xrightarrow{p} M$

$p^{-1}(x)$ = equivalence classes of maps $u \in U \to \mathbb{R}^n$

$0 = f(x) = g(x) \quad j^1_x f = j^1_x g$

$\Rightarrow$ $u$ defines isomorphism $T_xM$ with $\mathbb{R}^n = T_0\mathbb{R}^n$

$e_1, \dots, e_n$ standard basis of $\mathbb{R}^n \Rightarrow u^{-1}(e_1), \dots, u^{-1}(e_n)$ basis of $T_xM$

$\Rightarrow$ soldering form $\theta : TP^1M \to \mathbb{R}^n$

$$\theta(u)(\xi) = u(T_p(\xi))$$

$$\theta(ug) = Ad_{g^{-1}}\theta(u) \quad (\text{on } \mathbb{R}^n \times GL(n,\mathbb{R}))$$

$\operatorname{Ker}\theta \cong gl(n,\mathbb{R})$ spanned by fundamental vector field $\zeta_X$, $X \in gl(n,\mathbb{R})$

A connection on $P^1M$ $\quad \gamma : TP^1M \to gl(n,\mathbb{R})$

$$\gamma(\zeta_X) = X, \quad \underbrace{\gamma(ug) = Ad_{g^{-1}}\gamma(u)}_{\text{principal connection}}$$

Structure equations:

$d\theta = -[\theta,\gamma] - [\gamma,\theta] + T \leftarrow$ torsion of $\gamma$

$d\gamma = -[\gamma,\gamma] + R \leftarrow$ curvature of $\gamma$

... measure the failure of $\theta + \gamma$ be Maurer-Cartan form of $\mathbb{R}^n \times GL(n,\mathbb{R})$

Lemma: $T : \Lambda^2 TM \to TM$, $R : \Lambda^2 TM \to End(TM)$

or $T : P^1M \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$, $R : P^1M \to \Lambda^2\mathbb{R}^{n*} \otimes gl(n,\mathbb{R})$

Lemma: There is $\gamma$ such that $T = 0$ on $M$.

$G \subset GL(n,\mathbb{R})$ Lie subgroup

$\mathcal{G} \subset P^1M$ a $G$-structure if $\mathcal{G}$ locally trivialises

$\downarrow$ to $U \times G \subset U \times GL(n,\mathbb{R})$ over $U \subset M$

$M$ i.e. choice of $u: T_xM \to \mathbb{R}^n$ that differ by a gauge group $G$ in a smooth way

If $s: P^1M \to \mathbb{V}$ satisfies $s(ug^{-1}) = \varrho(g)s(u)$ for representation $\varrho$ of $GL(n,\mathbb{R})$ on $\mathbb{V}$

and $\forall x \in M\ \exists u \in P^1M$ $s(u) = N$, then $s^{-1}(N) \subset P^1M$ is a $G$-structure for $G = \{g \in GL(n,\mathbb{R}), \varrho(g)(N) = N\}$.

i.e. isotropy of $N$ in $GL(n,\mathbb{R})$.

Conversely, $\mathbb{V}$ decomposes to $\mathbb{V}_1 \oplus \dots \oplus \mathbb{V}_m$ ... representations of $G$, $N \in$ trivial representation define such $s: \mathcal{G} \to \mathbb{V}$, which extend to $s: P^1M \to \mathbb{V}$ by the rule $s(ug^{-1}) = \varrho(g)s(u)$.

Examples given by simple Lie groups

1) $\mathbb{V} = \Lambda^n \mathbb{R}^{n*}$, $s$ = volume form on $M$ for $N = e_1^* \wedge \dots \wedge e_n^*$

$G = SL(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}), \det(g) = 1\}$

2) $\mathbb{V} = S^2 \mathbb{R}^{n*}$, $s$ = metric on $M$ for $e_1^* \odot e_1^* + \dots + e_p^* \odot e_p^* - \dots - e_n^* \odot e_n^*$ of signature $(p, n-p)$

$G = O(p, n-p) = \{g \in GL(n,\mathbb{R}), g^T \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} g = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\}$

3) $\mathbb{V} = \Lambda^2 \mathbb{R}^{2n*}$, $s$ = almost symplectic form on $M$ for $e_1^* \wedge e_{n+1}^* + \dots$

$G = Sp(2n,\mathbb{R}) = \{g \in GL(2n,\mathbb{R}), g^T \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} g = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}\}$

4) $\mathbb{V} = \mathbb{C}^n \otimes \mathbb{C}^{n*}$, $s$ = almost complex structure on $M$ for $i \cdot id$

$G = GL(n,\mathbb{C}) = \{g \in GL(2n,\mathbb{R}), gi = ig\}$

5) $\mathbb{V} = S^2 \mathbb{C}^{n*}$, $s$ = complex metric, Re metric for complex basis)

$G = O(n,\mathbb{C}) = GL(n,\mathbb{C}) \cap O(n,n)$ $\quad g(a,a) = a^2$, $g(ia,ia) = -a^2$

6) $\mathbb{V} = \Lambda^2 \mathbb{C}^{n*}$, $s$ = complex almost symplectic form, $Re(-|-)$

$G = Sp(2n,\mathbb{C}) = GL(2n,\mathbb{C}) \cap Sp(4n,\mathbb{R})$

7) $\mathbb{V} = \mathbb{C}^{n*} \otimes \overline{\mathbb{C}^{n*}}$ $s$ = hermitian scalar product for $z_1\bar{z}_1 + \dots - z_p \bar{z}_p$ ...

$G = U(p,q) = \{g \in GL(n,\mathbb{C}), \bar{g}^T \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} g = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\}$

$\otimes^3 W = \mathbb{R}^{4m} \otimes \mathbb{R}^{4m*}$

8) $\mathbb{H}^n$ ← right quaternionic vector space, $i, j, k$ – complex structures

$s$ = hypercomplex structure on $M$ for $i, j, k$

$G = GL(n, \mathbb{H}) = \{ g \in GL(4n, \mathbb{R}),\ gi = ig,\ gj = jg,\ gk = kg \}$

9) $W \subset S^2 \mathbb{H}^{n*}$ $s$ = almost quaternionic Hermitian metrics

↳ compatible with $i, j, k$ $\quad Re(\bar{z}_1 w_1 + \dots - \bar{z}_{p+1} w_{p+1} - \dots)$

$G = Sp(p, q) = \{ g \in GL(n, \mathbb{H}),\ \bar{g}^T \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} g = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \}$

10) $W \subset \Lambda^2 \mathbb{H}^{n*}$ $s$ = almost quaternionic skew-Hermitian symplectic forms

↳ compatible with $i, j, k$ $\quad Re(\bar{z}_1 j z_1 + \dots \bar{z}_n j z_n \}$

$G = SO^*(2n) = \{ g \in GL(n, \mathbb{H}),\ \bar{g}^T j g = j \}$

11) Exceptional Lie groups $E_6, E_7, E_8, F_4, G_2$ ... not simple

Integrable G-structure $\mathcal{G} \subset P^1 M$

there are coordinates $\sigma_{U_i} : U_i \subset M \to \mathbb{R}^n$

such that $\mathcal{G} = U_i \times G$ for the corresponding trivialization

$\gamma : TU_i \times TG : \gamma(0_x, \xi_Z) = Z$ .. torsion free connection

it is a $G$ connection, i.e. $T\mathcal{G} \to \mathfrak{g}$, any other connection $\Omega^1(M, \mathfrak{g})$

$T = d\theta + [\gamma, \theta] + [\theta, \gamma]$ .. changes algebraically with $\Omega^1(M, \mathfrak{g})$

$T' - T = \delta A$ ... $A : \mathbb{R}^n \to \mathfrak{g}$, $\delta A(X, Y) = [A(X), A(Y)]$

Torsion of G-structure $T$ mod $\operatorname{Im} \delta$, vanishing ⟹

there is torsionfree connection

For some G-structures, this implies integrability

Example: $G = Sp(2n, \mathbb{R})$, Darboux theorem: $T = 0 \Leftrightarrow$ integrable

$G = GL(n, \mathbb{C})$, Newlander–Nirenberg: $T = 0 \Leftrightarrow$ integrable

$G = O(p, q)$, $T = 0, R = 0 \Leftrightarrow$ integrable

Automorphisms of G-structures: $\varphi : M \to M$, $P^1\varphi(\mathcal{G}) \subset \mathcal{G}$

1st-order G-structure $P_1\varphi(u) = P_1\psi(u) \Rightarrow \varphi = \psi$ for automorphisms

Theorem 1st-order G-structure $\Leftrightarrow$ $\operatorname{Ker} \delta = 0$

Homogeneous G-structures on homogeneous space $K/H$

$\Leftrightarrow$ $\alpha : \mathfrak{k}/\mathfrak{h} \to \mathbb{R}^n$ isomorphism induces $i : H \to GL(n, \mathbb{R})$

$\alpha(Ad(h)(X)) = Ad(i(h))\alpha(X)$, $i(H) \subset G \Rightarrow \mathcal{G} = K \times_{i(H)} G$

is G-structure for $\mathcal{G} \subset P^1 M$ induced by $\alpha \circ \omega_K$ $\quad K/H$