HOMOGENEOUS GEOMETRIC STRUCTURES

JAN GREGOROVIČ
UNIVERSITY OF HRADEC KRÁLOVÉ

SUMMER SCHOOL GEOMETRY AND TOPOLOGY

Lecture 2: Lie groups and algebras

4.1. Definition. A *Lie group* G is a smooth manifold and a group such that the multiplication $\mu : G \times G \to G$ is smooth. We shall see in a moment, that then also the inversion $\nu : G \to G$ turns out to be smooth.

We shall use the following notation:
$\mu : G \times G \to G$, multiplication, $\mu(x, y) = x.y$.
$\lambda_a : G \to G$, left translation, $\lambda_a(x) = a.x$.
$\rho_a : G \to G$, right translation, $\rho_a(x) = x.a$.
$\nu : G \to G$, inversion, $\nu(x) = x^{-1}$.
$e \in G$, the unit element.

$\mu(x, \nu(x)) = e$, $T_e\nu_e = id$ + implicit function Thm $\Rightarrow$ smooth

$T_{(a,b)}\mu(X_a, Y_b) : T_aG \times T_bG \to T_{ab}G$, $a, b \in G$
$= T_a\rho_b(X_a) + T_b\lambda_a(Y_b)$

Example: $GL(2,\mathbb{R}) = \left\{\begin{pmatrix} a & c \\ b & d\end{pmatrix}, a\cdot d - b\cdot c \neq 0\right\} \subset \mathbb{R}^4$

Another coordinates $= \left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\cdot\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix}\cdot\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}, x\cdot y \neq 0\right\}$
$\subset [0,2\pi] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$

Exercise: what is the transition map?

Example: $SL(2,\mathbb{R}) = \left\{\begin{pmatrix} a & c \\ b & d\end{pmatrix}, a\cdot d - b\cdot c = 1\right\}$
map from $\mathbb{R}^3$: $(a, b, c) \to \begin{pmatrix} a & b \\ c & \frac{1+bc}{a}\end{pmatrix}$

Exercise: map covering $\begin{pmatrix} 0 & b \\ c & d\end{pmatrix}$, $-bc = 1$?
Another map $(\theta, x, z) \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & \frac{1}{x}\end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}$

Example: $O(2) = \{A \in GL(2) \quad A^T = A^{-1}\}$

Exercise: Check $\theta \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ is map $\mathbb{R}$ onto $SO(2)$

Maurer-Cartan form $\omega_G \in \Omega^1(G, T_eG)$

$\omega_G(h)(\underset{T_hG}{X_h}) = \underset{T_eG}{T_h\lambda_h^{-1}(X_h)}$ – 1-form with values in T_eG

Left invariant vector field $L_X := \omega_G^{-1}(X)$, $X \in T_eG$

i.e. $L_X(h) = T_e\lambda_h(X) \Rightarrow TG = G \times T_eG$ ... trivial bundle

Lie algebra $\mathfrak{g}$ of G is T_eG together with

Lie bracket $[X,Y] := -d\omega_G(L_X, L_Y)$

Lemma $d \circ d = 0$ implies Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Proof: Exercise.

Adjoint representation $Ad : G \to GL(\mathfrak{g})$

$$Ad(g)(Y) := \omega_G(T_e g^{-1} Y)$$

$T_eAd = ad : \mathfrak{g} \to End(\mathfrak{g})$

$$ad(X)(Y) = T_eAd(X)(Y)$$

Exercise: check $ad(X)(Y) = [X,Y]$

Examples: $gl(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right\}$ ... trace $tr\begin{pmatrix} a & c \\ b & d \end{pmatrix} = a+d$

$T_e\det : gl(2,\mathbb{R}) \to \mathbb{R}$, $\frac{d}{dt}\begin{pmatrix} a(t) & c(t) \\ b(t) & d(t) \end{pmatrix} = a \cdot 1 + 1 \cdot d - b \cdot 0 - 0 \cdot c$ ✓

$sl(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & c \\ b & -a \end{pmatrix} \right\}$

$A^T = A^{-1} \Rightarrow A^T(t) \cdot A(t) = id \Rightarrow A^T \cdot id + id^T \cdot A = 0$

$so(2) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \right\}$

Theorem (Ado) For "every" Lie algebra there is $G \subset GL(n,\mathbb{R})$ Lie subgroup with this Lie algebra.

Logarithmic (Darboux) derivative of $f : M \to G$

$\omega_f : TM \to \mathfrak{g}$ $\quad \omega_f(X) = \omega_G(Tf(X))$

Theorem (Cartan) The following is equivalent for $\omega : TM \to \mathfrak{g}$:

1) $d\omega(\xi,\eta) = -[\omega(\xi), \omega(\eta)]$ (Maurer-Cartan equation)

2) There is atlas $\varphi_i : U_i \to G$ such that $\omega = \omega_{\varphi_i}$ and the transition maps are left multiplication by elements of G

Exercise: for $G = (\mathbb{R},+)$ show this is fundamental theorem of calculus

$$\begin{array}{ccc} & S^1 & \mathbb{RP}^1 \\ & \cong & \cong \end{array}$$

One-dimensional subgroups, $\mathbb{R}$, $SO(2)$, $PSO(2)$

$c(t)\cdot c(s) = c(t+s)$, $c(0)=e$ $\quad (c(t+p)=c(t))$

Lemma: There is bijection between one dimensional sub-groups of G and open subset of $\mathfrak{g}$.

Proof: Exponential map $\exp(A)$ solution of $\omega_G(\dot c(t)) = A$ at time 1

Diffeomorphism of open subset of $\mathfrak{g}$ to G

$\exp(sA)\exp(tA) = \exp((s+t)A)$

by uniqueness of solutions this shows that $\exp(tX)$ is defined for all t and a one dimensional subgroup, i.e., LX is complete vector field $\square$

$\Rightarrow$ Exponential coordinates $\ln: U\subset G \to \mathfrak{g}$ inverse of $\exp$

In general, $\exp$ is not injective nor surjective $\Rightarrow$ use left multiplication to cover G by an atlas of exponential coordinates

Baker-Campbell-Hausdorff formula

$\ln(\exp(X)\cdot\exp(Y)) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[Y,X]])$

$+ \dots$ (in general infinite length, has to converge)

Nilpotent Lie algebra $[[X_1 \dots, [X_i, Y]] = 0 \quad \forall X_1 \dots X_i, Y \in \mathfrak{g}$

$\Rightarrow$ BCH formula is globally defined

Representation $\rho: G \to GL(V)$

$d\rho: \mathfrak{g} \to End(V)$ ... Lie algebra representation

$d\rho(X)d\rho(Y)(v) - d\rho(Y)d\rho(X)(v) = d\rho([X,Y])(v)$

Lemma $\rho(\exp(X))(v) = \exp^{d\rho(X)}(v)$

Over complex numbers $d\rho(X) = S + N$ ... Jordan decomposition for S diagonal and N nilpotent. Ex: $\overset{S}{\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}} + \overset{N}{\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}}$ ... Jordan block

in some basis of $V \Rightarrow \exp^{d\rho(X)} = T\begin{pmatrix} e^a & 0 \\ 0 & e^a\end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}T^{-1}$

T - transition matrix

! $\exp(X) = id$ does not imply $\exp^{d\rho(X)} = id$ ... ρ does not have to exist for $d\rho$

Lie subgroup H ... inclusion $H \to G$ is smooth

Lie subalgebra $\underline{h}$ ... $h \subset g$, $[h,h] \subset h$

Lemma For every Lie subalgebra h, there is Lie subgroup H with Lie algebra $\underline{h}$.

But does not have to be closed - infinite spiral on $S^1 \times S^1$

Representation $\rho : G \to GL(V)$, isotropy G_v of v is closed Lie subgroup $\{w, \rho(w)(v) = v\}$

isotropy subalgebra is ann(v) $\{X, d\rho(X)(v) = 0\}$ of g

$\rho(G)v$ ... G-orbit of v, $G_w \cong G_v$ for $w \in Gv$

isomorphic to G/G_v + homogeneous space by

Theorem. *If H is a closed subgroup of G, then there exists a unique structure of a smooth manifold on G/H such that $p : G \to G/H$ is a submersion. So* $\dim G/H = \dim G - \dim H$.

$U \times H \subset G$

There is local trivialisation $U \subset G/H$

$\Rightarrow$ $G \to G/H$ is principal H bundle

right action $G \times H \to G$ $r_h(g) = g \cdot h$

Fundamental vektorfield $\zeta_X(g) = T_{(g,e)} r \cdot (0_g, X)$

Example: $Ad : GL(2,\mathbb{R}) \to gl(2,\mathbb{R})$: $Ad(X)(Y) = X \cdot Y \cdot X^{-1}$

$GL(2,\mathbb{R})$ orbits = Jordan normal forms up to reordering of Jordan blocks

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a < b$, $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

Exercise: $\mathrm{ann}(\quad) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\}$, $\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$, $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ $b = 0$

$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ $b \neq 0$

Exercise: are there any elements of G_v not in the image of exponential map of above