HOMOGENEOUS GEOMETRIC STRUCTURES

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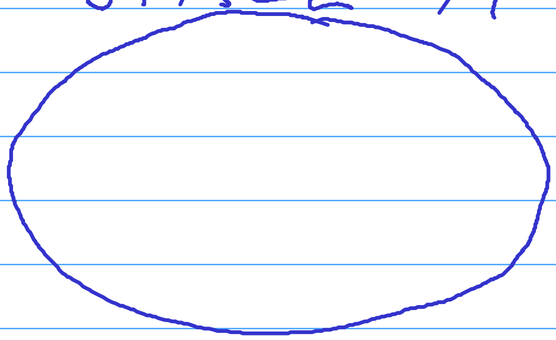
SUMMER SCHOOL GEOMETRY AND TOPOLOGY

Plan of the lectures:

1) Introduction
2) Lie Groups and algebras
3) G-structures
4) Symmetric spaces
5) Cartan geometries

Introduction:

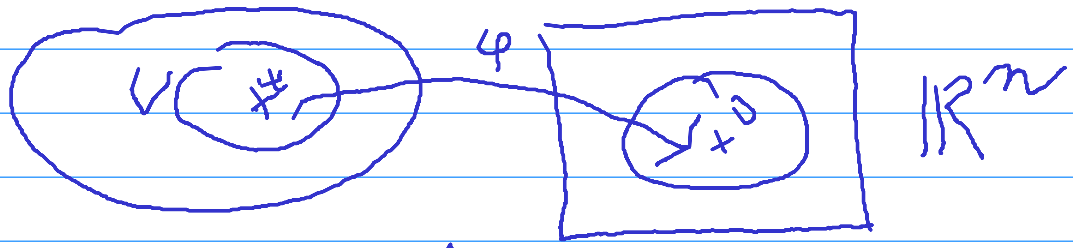
SPACE M



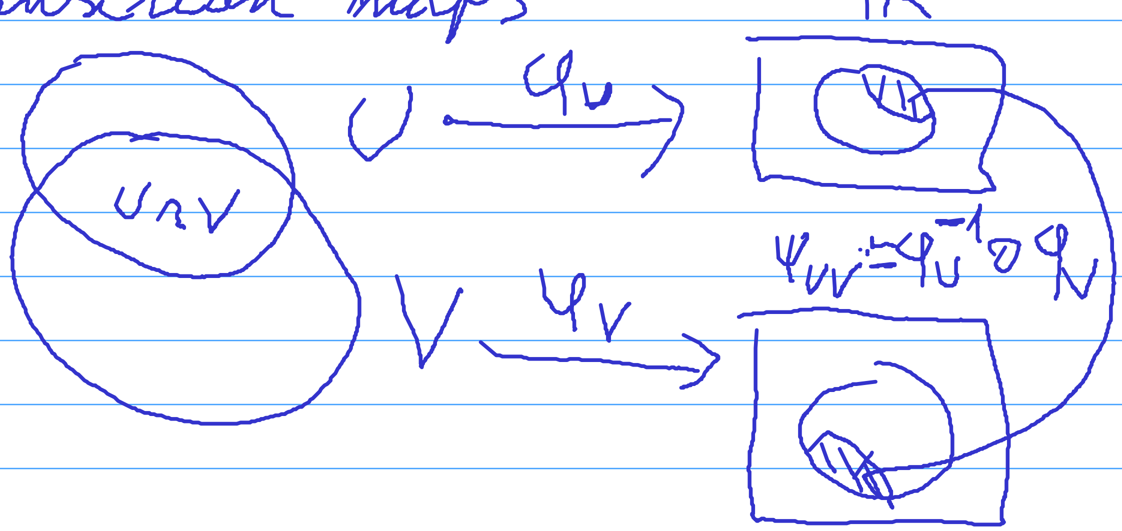
+ geometric structure
  - allows compute something
  - allows measure something

- topological space - collection of open sets U
- countable basis V_i covering M
  = every $x \in M$ belongs to some V_i
- Hausdorff:
  $x \in U$ neighborhood of x = collection of points "close" to x
  x and y have neighborhoods U, V $U \cap V = \emptyset$

How to compute something on M?
- choose local coordinates



φ: homeomorphism U onto open subset of $\mathbb{R}^n$
= open subsets go onto open subsets
$\varphi(x) = 0$ — centered at $x \in U$
- Atlas = collection of all maps
- transition maps



does ψ_{UV} preserve what you computed?
Transition maps form a pseudo group on $\mathbb{R}^n$
= satisfy group relations if the compositions are defined

(Integrable) geometric structures
→ special choice of coordinates such that transition maps preserve what you want to compute
- choose a pseudogroup on $\mathbb{R}^n$ and assume that transition maps belong to it

Examples of pseudo-groups

- Homogeneity is required $= \forall x, y \in \mathbb{R}^n$ there is ϕ in the pseudo group s.t. $\phi(x) = y$

1) Mappings of class C^r
   $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is r-times
   $\mapsto (f_1, \dots, f_n)$ continuously differentiable
   C^0: continuous functions
   C^∞: smooth functions
   C^0-atlas = all transitions are C^0
   = topological manifolds
   C^∞-atlas = all transitions are C^∞
   = smooth manifolds

2) Real analytic maps
   - Taylor expansions of f and g at x coincide in all orders $\Rightarrow f = g$
   C^ω functions = real analytic functions
   C^ω atlas = all transitions are C^ω
   = real analytic manifold

3) Holomorphic maps
   $f : U \subset \mathbb{C}^N \longrightarrow \mathbb{C}^N$
   satisfies CR equations $\bar{\partial} f = 0$
   $\Rightarrow f$ is power series in complex variable
   holomorphic atlas = all transitions are holomorphic = complex manifolds

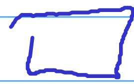
4) Maps preserving
   - Tensors
   - Orientation
   - ...

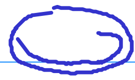
GENERAL IDEA = USE HOMOGENEOUS GEOMETRIC STRUCTURES TO DEFINE GEOMETRIC STRUCTURES

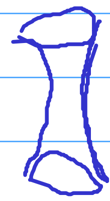
= F. KLEIN'S ERLANGEN PROGRAM

- E. CARTAN connected it with diff. geometry (LECTURE 5)

- above structures are flat
  $\Rightarrow$ use of homogeneous structures allow some nonflatness

Example: $\mathbb{R}^n$ ... flat euclidean space ... 

S^n ... round sphere 

H^n ... hyperbolic space 

Symmetric spaces (LECTURE 4)

Relations between geometric structures

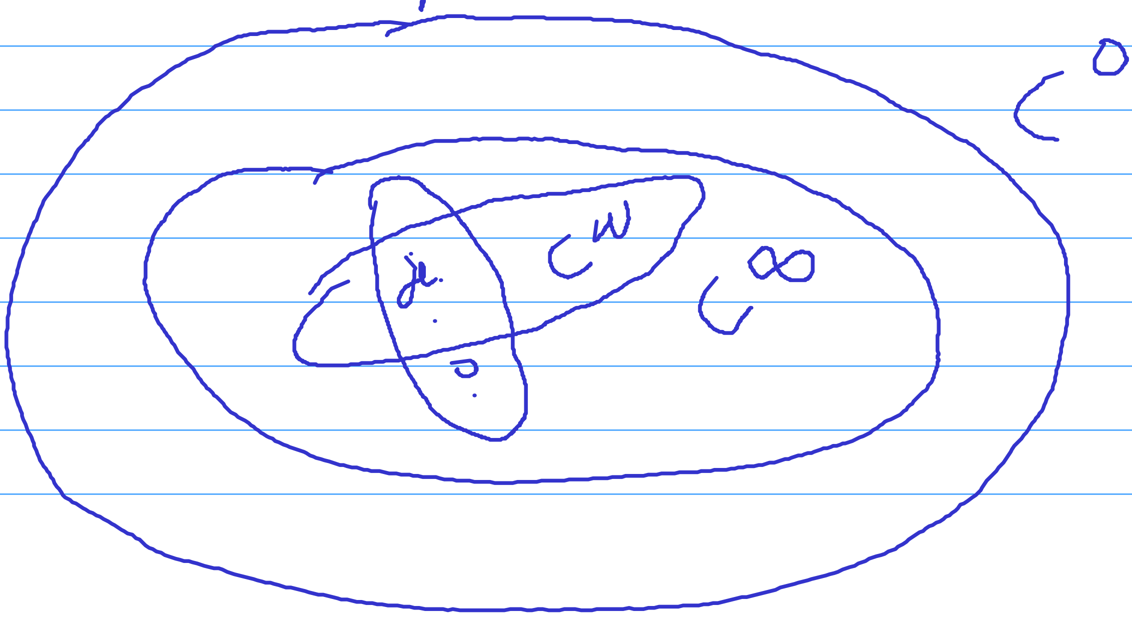
$C^k \Rightarrow C^l \quad l < k$

real analytic $\Rightarrow C^\infty$

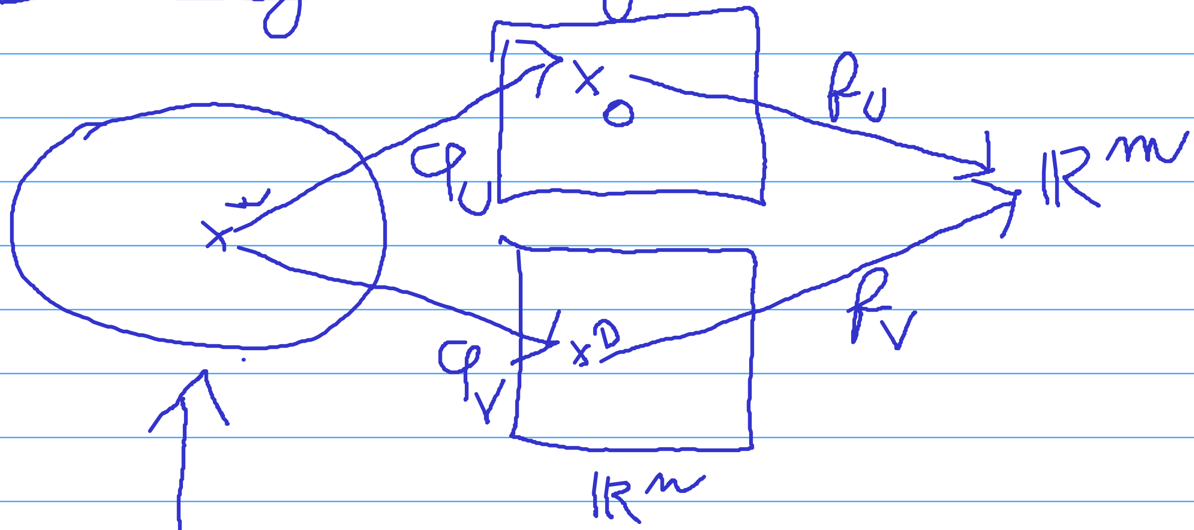
holomorphic $\Rightarrow C^\omega$

"elliptic" geometric structures $\Rightarrow C^\infty$

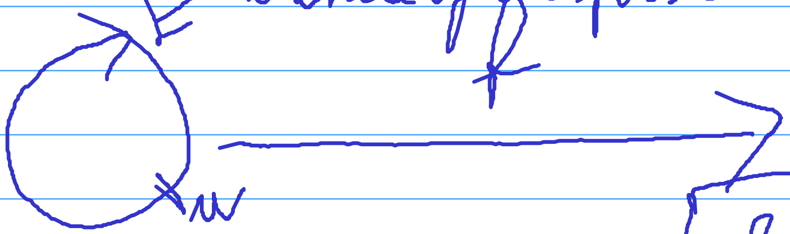
usually one assumes geometric structure + one from above



Measuring something



"bundle" of all possible coordinates



Relative invariant

$$f(ug^{-1}) = S_g f(u)$$

$g \in G_0$

G_0 – group generated by transition maps fixing 0 = gauge group

S – representation of G_0 on $\mathbb{R}^m$

absolute invariant $f(ug^{-1}) = f(u)$

fix a normal form $N \in \mathbb{R}^m$

$f^{-1}(N)$ ... select a sub (pseudo) group

$\Rightarrow$ defines another geom. structure

(Lecture 3)

Problem: $G_0 = \mathrm{Diff}_0$ on smooth mfd is too big for actual computation

In particular contains maps that are id on U but nontrivial outside of U

Jets: $j^k_0 f = j^k_0 g$ if Taylor expansions of f and g coincide up to order k at 0

$\Rightarrow$ relation of equivalence $\sim_k$

k-th order frame bundle

$P^k M =$ "bundle of all coordinates" $/\sim_k$

$P^0 M = M$

$P^1 M =$ bundle with gauge group $GL(n, \mathbb{R})$

$=$ Jacobi matrices of $f: U \subset \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = x$

$J_x(f)$

In local coords: $\mathbb{R}^n \rtimes GL(n, \mathbb{R})$ .. semi-direct product, $GL(n,\mathbb{R})$ acting on $\mathbb{R}^n$

. choose of representatives for the equivalence class

$\Rightarrow$ local trivialisation $U \times GL(n, \mathbb{R})$

$R_g(x, h) = (x, hg)$ ... right action of $GL(n, \mathbb{R})$

$\Rightarrow$ choose an atlas of local trivialisations

$\varphi_U: U \to P^1 M = \varphi_U(U) \times GL(n, \mathbb{R})$

Different atlas provides function $\varphi_{UV}: U \cap V \to G_0$

$\varphi_U(x) = \varphi_V(x)\, \varphi_{UV}(x)$

$\varphi_{UW} = \varphi_{UV}\, \varphi_{VW}$ ~ cocycle condition

$P^k M$ .. gauge group G^k_n ... analogously

NATURAL bundles ... $\mathbb{V}$ representation of G^k_n

$$P^k M \times_{G^k_n} \mathbb{V} := P^k M \times \mathbb{V} / (u g^{-1}, g v) \sim (u, v) \quad g \in G^k_n$$

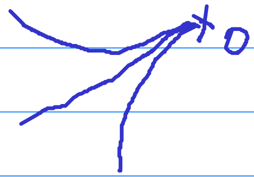
... associated bundle, where relative invariants of smooth manifolds live

Example:

$TM := P^1M \times_{GL(n,\mathbb{R})} \mathbb{R}^n$ .. tangent bundle

$c: \mathbb{R} \to M$ curve $TM \cong$ space of curves$/\sim_1$

= directions on smooth manifold



$T^*M := P^1M \times_{GL(n,\mathbb{R})} (\mathbb{R}^n)^*$ ... cotangent bundle

smooth $M \to \mathbb{R}$ function $T^*M \cong$ space of functions$/\sim_1$
$C^\infty(M,\mathbb{R})$

TENSOR bundles $P^1M \times_{GL(n,\mathbb{R})} (\mathbb{R}^n)^{\otimes k} \otimes (\mathbb{R}^{n*})^{\otimes l}$

symmetric, antisymmetric $P^1M \times_{GL(n,\mathbb{R})} S^k\mathbb{R}^n \otimes S^l(\mathbb{R}^{n*})$

$S^kTM \otimes S^lT^*M \otimes \Lambda^kTM \otimes \Lambda^lT^*M$ $\quad \Lambda^k\mathbb{R}^n \otimes \Lambda^l(\mathbb{R}^{n*})$

Sections of natural bundles = G^k_n-equivariant functions
$P^kM \to V$

Ω^1M = sections of T^*M – 1-forms

$\Gamma(TM)$ = sections of TM = vector fields

Ω^kM = sections of Λ^kT^*M = k-forms

Directional derivative $\Gamma(TM) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$

$X.f(x) = \frac{d}{dt}\Big|_{t=0} f(c_x(t))$ for $\frac{d}{dt}\Big|_{t=0} c_x(t) = X(x)$
$c_x: \mathbb{R} \to M$

Lie Bracket $[X,Y]f = X.(Y.f) - Y.(X.f)$

Differential

$d: \Omega^{k-1}M \to \Omega^kM$, generalises differential of func
$d: C^\infty(M,\mathbb{R}) \to \Omega^1(M)$

induced by directional derivative

$$dw(\xi_0,\dots,\xi_k) = \sum (-1)^i \xi_i\, w(\xi_0,\dots,\hat{\xi_i},\dots,\xi_k) + \sum (-1)^{i+j} w([\xi_i,\xi_j],\xi_0,\dots,\xi_k)$$

$d \circ d = 0$

DeRham sequence

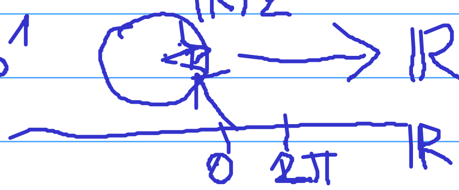
$$0 \to C^\infty(M,\mathbb{R}) \xrightarrow{d} \Omega^1(M) \to \dots \xrightarrow{d} \Omega^n(M) \to 0$$

$\downarrow \qquad \downarrow \qquad \downarrow$

$H^i := \operatorname{Ker} d / \operatorname{Im} d \qquad H^n$

- captures information about topology of M

Example: S^1 $\mathbb{R}/\mathbb{Z}$ $\to \mathbb{R}$ $\quad f(2\pi + k) = f(k)$ periodic



1-form = volume form μ

$$\int_{S_1} \mu = 0 \iff \mu = df$$

$[\mu] \in H^1$ obstruction for finding f s.t. $\mu = df$

Distribution: annihilators Θ^i - lin independent $\} \subset \Omega^1(M)$

I = ideal in $\Omega^*(M)$ spanned by

Frobenius: $dI \subset I \iff \Theta^i$ annihilate TN of integral submanifold $N \subseteq M$ through x for $\forall x$

$\Rightarrow$ Higher cohomology play a global obstruction

H^n - obstruction for using Stokes theorem

$$\int_{\partial\Omega} \omega = \int_\Omega d\omega$$

Whitney embedding: $M \to \mathbb{R}^{2n}$

H^i provides obstruction for embedding to smaller $\mathbb{R}^{n'}$