Lecture 5

<u>Problem</u>: classify (connected) holonomy groups of pseudo-Riemannian mfs.

- Riemannian mfs.
  Berger '53, ..., Bryant '89
- Lorentzian mfs $(1, n-1)$
  $e_1, e_2 \ldots, e_n$ $(e_1, e_1) = -1$
  $(e_i, e_i) = 1$, $i \geq 2, \ldots$
  Berard-Bergery, Ikemakhen '93
  Leistner '03
  Galaev '05
- other signatures, e.g. $(2, n-2)$
  not solved, partial results

- Riemannian:

A. Besse Einstein manifolds, 1987
Chapter 10

Alekseevsky 1990   Books Joyce

- A. Galaev Holonomy groups of Lorentzian mfds
Russian Math. Reviews 2015

- Galaev, Leistner, Recent developments in holonomy ~~theory~~ 2009?

- references

Galaev, Hol. classif of Lorentz-Kähler
J. Geom. Anal 2019

(M,g) , (N,h)

$$(M\times N\,,\ \overbrace{g+h}^{=B})$$

$$T_{(x,y)}(M\times N) = \underbrace{T_xM}_{g_x} \overset{\perp}{\oplus} \underbrace{T_yN}_{h_y}$$

$$Hol_{(x,y)}(M\times N) = Hol_x(M)\times Hol_y(N) =$$

$$= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \ \middle|\ \begin{matrix} A\in Hol_x(M) \\ B\in Hol_y(N) \end{matrix} \right\}$$

$$\gamma:[a,b]\to M\times N \Rightarrow \gamma(t) = (\gamma_1(t), \gamma_2(t))$$

$$\underbrace{\underbrace{x^1,\dots,x^n}_{i},\ \underbrace{y^1,\dots,y^m}_{\alpha}}_{a}$$

$$B\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha}\right)=0$$

$$B_{ij} = g_{ij}$$

$$B_{\alpha\beta} = h_{\alpha\beta}$$

$$X = X^a \frac{\partial}{\partial z^a}\Big|_{\gamma(t)}$$

Γ^i_{jk}

$\Gamma^\alpha_{\beta\gamma}$

$$\dot X^a(t) + X^b(t)\,\dot\gamma^c(t)\,\Gamma^a_{cb} = 0 \quad \forall a$$

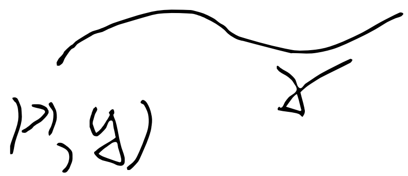
$$\dot X^i(t) + X^j\dot\gamma^k\Gamma^i_{jk} = 0 \quad \forall i$$

$$\dot X^\alpha(t) + X^\gamma\dot\gamma^\beta\Gamma^\alpha_{\gamma\beta} = 0 \quad \forall \alpha$$

$$X_0 = Y_0 + Z_0$$

$$\uparrow \qquad\qquad \uparrow$$

$$T_{\gamma_1} M \qquad T_{\gamma_2} M)$$



$$\tau_\gamma X_0 = \underbrace{\tau_{\gamma_1} Y_0}_{\text{on } M} + \tau_{\gamma_2} Z_0$$

$$\tau_\gamma = \left(\begin{array}{c|c} \tau_{\gamma_1} & 0 \\ \hline 0 & \tau_{\gamma_2} \end{array}\right) \qquad \square$$

(M, g) is Riem.

$\mathrm{Hol}_x(g)$ preser. $V_x \subset T_x M$

$\forall A \in \mathrm{Hol}_x(g), X_x \in V_x \quad A X_x \in V_x$

$V_x^{\perp} \quad \mathrm{Hol}_x(g)$ pres. $V_x^{\perp}$

$Y_x \in V_x^{\perp}$

$$g_x(AY_x, X_x) = g_x(Y_x, \underbrace{A^{-1}X_x}_{\in V_x}) = 0$$

$X_x \in V$

$$\Rightarrow A Y_x \in V_x^{\perp}$$

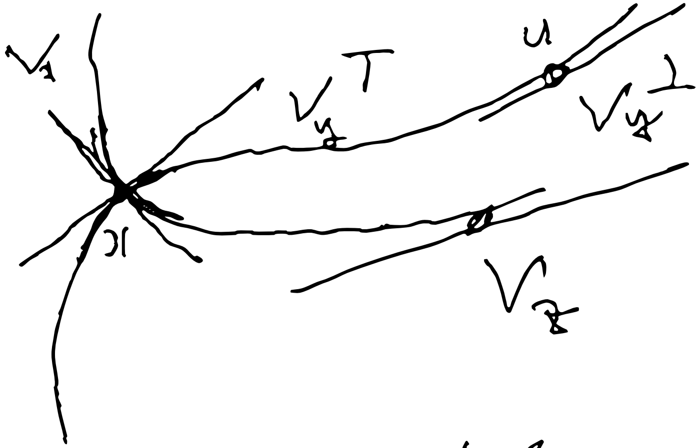
$$T_x M = V_x \overset{\perp}{\oplus} V_x^{\perp}$$

$\Rightarrow V, V^{\perp}$ ∥ distri.

$X \in \Gamma(V) \qquad \nabla_Z X \in \Gamma(V) \quad \forall Z$

V is involutive: $X, Y \in \Gamma(V)$

$$[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(V)$$



$\exists\, U \ni x$

$U = U_1 \times U_2$

$g = g_1 + g_2$

Borel-Lichnerowicz Th. (de Rham)

M, $\pi_1(M) = 0$, complete

$\exists\, V$

$\Rightarrow M = M_1 \times M_2$

$g = g_1 + g_2$

<u>Reduction</u> We may assume that $Hol_x(g)$ is <u>irreducible</u>

Not true for ps-Riem case!

Lor. signature $\mathbb{R}^{1,2}$

e_-, e_1, e_+

$g(e_-, e_-) = -1$

$g(e_1, e_1) = 1$

$g(e_+, e_+) = 1$

$p = \frac{\sqrt{2}}{2}(e_- - e_+)$

$q = \frac{\sqrt{2}}{2}(e_- + e_+)$

$g(p, p) = g(q, q) = 0$

$g(p, q) = -1$

p, e_1, q

$G \subset O(1,2)$

Preserve $\mathbb{R}p$

$(\mathbb{R}p)^\perp = \langle p, e_1 \rangle$

$~~\mathbb{R}^{1,2} = (\mathbb{R}p) \oplus (\mathbb{R}p)^\perp~~$

Riem. mf.

<u>Th.</u> (M, g) is anal.

$$\mathrm{hol}_x(g) = \mathrm{span}\{ R_x(X, Y),\ \nabla_{X_1}(R(X, Y))|_x, \ldots \ X, Y, X_1, \ldots \in T_x M \}$$

(M, g) locally symmetric

$$\nabla R = 0$$

$$\mathrm{hol}_x(g) = \mathrm{span}\{ R_x(X, Y) \mid X, Y \in T_x M \}$$

if M is simply connected, symmetric

$\Rightarrow M$ can be reconstructed ~~from~~ from $\mathrm{hol}_x(g)$, R_x)

$$\mathfrak{g}_0 := \mathrm{hol}_x(D) \qquad \mathfrak{g}_1 := T_x M$$

$$\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$$A, B \in \mathfrak{g}_0 \qquad [A, B] = [A, B]_{\mathfrak{g}_0}$$

$$A \in \mathfrak{g}_0, \quad X \in \mathfrak{g}_1, \quad [A, X] := AX$$

$$[X, Y] := -R_x(X, Y) \in \mathfrak{g}_0$$

$$\mathfrak{g}_0 \cdot R = 0$$

$$[A, R(X, Y)] = R(AX, Y) + R(X, AY)$$

$$G/G_0 = M$$

assume $\nabla R \neq 0$

$\exp : \mathfrak{g} \to G$

$\dim M = n$

<u>Th.</u> (M,g) indecomp. (hol gr. is irr.)

$\nabla R \neq 0$, $\pi_1(M) = 0$

$\Rightarrow$ $Hol(g)$		Einstein	$\parallel$ spinor
$SO(n)$		?	—
$U(m)$	$n = 2m$	?	—
$SU(m)$		$Ric = 0$	✓
$Sp(k)$	$n = 4k$	$Ric = 0$	✓
$Sp(k) \cdot Sp(1)$		$Ric = \lambda g$, $\lambda \neq 0$	—
G_2	$n = 7$	$Ric = 0$	✓
$Spin(7)$	$n = 8$	$Ric = 0$	✓

$n = 2m \qquad \mathbb{R}^{2m} = \mathbb{C}^m \qquad J_x : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$

$$U(m) = \{ A \in SO(2m) \mid A J_x = J_x A \}$$

$g_x(J_x X_x, J_x Y_x) = g_x(X, Y)$

$J_x^2 = -\mathrm{id}$

$\Longleftrightarrow \exists J \quad \nabla J = 0$

Kähler

$$SU(m) = \{ A \in U(m) \mid \det_{\mathbb{C}} A = 1 \}$$

$A : \mathbb{C}^m \to \mathbb{C}^m$

special Kähler

Calabi-Yau <u>mf.</u>

$(M, g) \rightsquigarrow R \qquad R(X_x, Y_x) Z_x \in T_x M$

$$Ric_x(X_x, Z_x) = \mathrm{tr}\,(\underbrace{Y_x}_{T_x M} \mapsto \underbrace{R(X_x, Y_x) Z_x}_{T_x M})$$

$$Ric_{ij} = R^k_{ijk}$$

(M,g) is C.-Y. $\Leftrightarrow$ (M,g) is Kähler,
$\underline{Ric = 0}$

Einstein eq:
(M,g) is Einstein
$Ric = \Lambda g$ $\quad \Lambda \in \mathbb{R}$

$M = \mathbb{R}^{1,3} \times N^n$ $\quad n = 7$
compact $\quad G_2$

$n = 6$
$SU(3)$