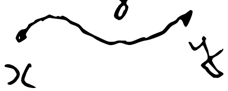
Lecture 4

$$\begin{tikzcd} E \arrow[d] \\ M \end{tikzcd}$$

∇



$\tau_\gamma : E_x \to E_y$

$\nabla \xi = 0$

$\mathrm{Hol}_x(\nabla)$

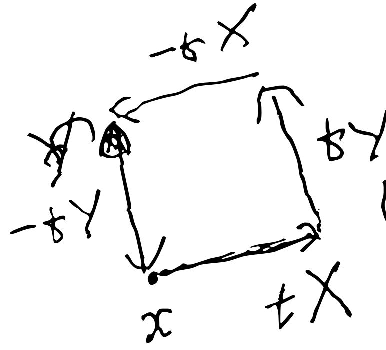
$E \to M$ $\quad\nabla\quad$ $R(X,Y)\xi := \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi$ $\in \Gamma(E)$

$X, Y \in \Gamma(TM)$, $\xi \in \Gamma(E)$

$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$

$C^\infty M$- multilinear $\quad x \in R \quad$ $R_x : T_xM \times T_xM \times E_x \to E_x$

$R_x(X_x, Y_x)\xi_x$



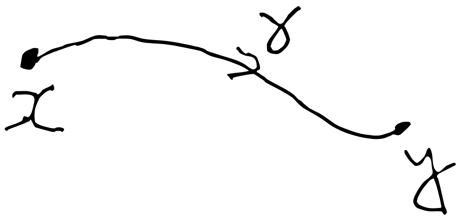
$$\underbrace{\frac{d}{dt}\left(-\tau_{tY} \circ \tau_{-tX} \circ \tau_{tY} \circ \tau_{tX}\right)\Big|_{t=0}}_{R_x(X,Y)} \in \mathfrak{hol}_x(\nabla)$$

$X, Y \in T_xM$

$R(X,Y) : E_x \to E_x$

Th. (Ambrose - Singer)

$$\mathrm{hol}_x(\nabla) = \mathrm{span}\Big\{ \underbrace{\tau_\gamma^{-1}\circ \overbrace{R_y(X,Y)}^{E_y\to E_y}\circ\tau_\gamma} \;\Big|$$



$$\Big\}$$

$\tau_\gamma : E_x \to E_y$

Def ∇ is called <u>flat</u> if $\forall\, x \in U$

$\exists\ \xi_1,\dots,\xi_m \in \Gamma(E|_U)$ $\quad \nabla\xi_\alpha = 0$

Th. $\underset{(1)}{\nabla \text{ is flat}} \iff \underset{(2)}{R = 0} \iff \underset{(3)}{\mathrm{hol}_x(\nabla) = 0} \iff \underset{(4)}{\mathrm{Hol}^0_x(\nabla) = \{id\}}$

$(2)\iff(3)$ A-S. Th.

$(3)\iff(4)$ Lie Th.

$(1)\Rightarrow(2)$ $\quad R(X,Y)\xi_\alpha = 0 \quad \Rightarrow R(X,Y) = 0 \Rightarrow R = 0$

$(4)\Rightarrow(1)$ $\quad \mathrm{Hol}^0_x(\nabla) = \{id\}$ $\quad \underset{\substack{\downarrow \\ x}}{U}$ $\quad \mathrm{Hol}_x(\nabla|_U) = \{id\}$

$\xi_{1x},\dots,\xi_{mx} \in E_x$

$\Rightarrow \exists\ \xi_1\dots\xi_m \quad \nabla\xi_\alpha = 0$ $\quad\square$

M ∇ on TM

∇ is defined $T^{p,q}M$

V Be a vector space

$(1,0)$	$X \in V$	
$(0,1)$	$\theta \in V^*$	$\theta(Y) \in \mathbb{R}$
$(0,2)$	$B : V \times V \to \mathbb{R}$	
$(1,2)$	$B : V \times V \to V$	
$(1,1)$	$A : V \to V$	$A(X) \in V$

T_xM $T_x^{(p,q)}M$

T_x^*M $T^{(p,q)}M = \bigcup_{x \in M} T_x^{(p,q)}M$

vector bundle over M

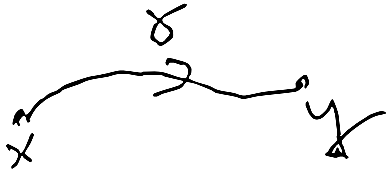
φ $(1,1)$ $\forall x \in M$ $\varphi_x : T_xM \to T_xM$

ω $(0,1)$ $\forall x \in M$ $\omega_x : T_xM \to \mathbb{R}$

A tensor field (p,q) $\nabla_X A$

$(0,1)$ $\theta \in \Gamma(T^*M)$ $(\nabla_X \theta)(Y) :=$

$:= X(\theta(Y)) - \theta(\nabla_X Y)$



$$\underbrace{(\tau_\gamma \theta_x)}_{\in T_y^* M}(X_y) = \theta_x(\tau_\gamma^{-1} X_y) \in \mathbb{R}$$

(M, g) g $(0,2)$ $\forall x \in M$ $g_x : T_xM \times T_xM \to \mathbb{R}$

symmetric, bil.

<u>positive - definite</u>

$g_x(X_x, X_x) > 0$, $X_x \neq 0$

(pseud-Riem.) non-degenerate

$$(\nabla_X g)(Y, Z) = X(\underbrace{g(Y, Z)}_{}) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

torsion $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

$X, Y \in \Gamma(TM)$

T is def only for con. on TM

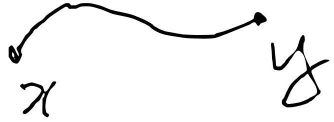
$$T(fX, Y) = f\,T(X,Y)$$

∇ not a tensor field

Th. (M, g) $\exists!\ \nabla$

1. $\nabla g = 0$
2. $T = 0$

Levi-Civita connection



$X, Y \in T_xM$

$$\tau_\gamma \, g_x = g_y$$

$$g_x(X, Y) = g_y(\tau_\gamma X, \tau_\gamma Y)$$

$$g_x(\tau_\gamma X, \tau_\gamma Y) = g_x(X, Y)$$

$$\Rightarrow \tau_\gamma \in O(T_xM, g_x) \cong O(n)$$

$$O(T_xM, g_x) = \{ a : T_xM \to T_xM \mid g_x(aX, aY) \} = g_x(X, Y) \}$$

$$Hol_x(g) \subset O(T_xM, g_x) \cong O(n)$$

$\mathfrak{g}$ Lie alg $\quad \mathfrak{g} \to \mathfrak{gl}(V)$

$\mathfrak{g}$ on $\bigotimes^{(p,q)} V$

V^* $\quad A \in \mathfrak{g} \quad A : V \to V$

$$(A\theta)(X) = -\theta(AX)$$

$(0,2)$ $B : V \times V \to \mathbb{R}$ $\quad (A\cdot B)(X,Y) =$

$$= -B(AX,Y) - B(X,AY)$$

$\underset{\langle \cdot,\cdot \rangle}{B}$ is scalar pr on $\mathbb{R}^n$ $\quad A \in \mathfrak{o}(n) \iff A \cdot B = B$

Cor. (M, ∇)

$\{A$ T.f. ~~of~~ (p,q), $\nabla A\} \iff \{A_x \in \bigotimes^{(p,q)} T_x M$

$\alpha \cdot A_x = A_x$

$\forall\, \alpha \in \mathrm{Hol}_x(\nabla)\}$

Prop. (M,g) Riem. mf.

M is orientable $\Leftrightarrow$ $\mathrm{Hol}_x(g) \subset SO(n)$

$$SO(n) = \{A \in O(n) \mid \det A = 1\}$$

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2)$

$A : \mathbb{R}^n \to \mathbb{R}^n$

$\det A = -1$

M is orientable: $\mathcal{A}$ on M $\quad U_i\ U_j$

$\varphi_{ij} : U_{ij} \to U_{ji}$

$\Leftrightarrow$ $\exists$ non-vanishing n-form

$$\omega_x : \underbrace{T_xM \times \cdots \times T_xM}_{n \text{ times}} \to \mathbb{R}$$

$$\omega(X_1, X_2, \dots) = -\omega(X_2, X_1, \dots)$$

M is oriented $\mathrm{Vol} := \sqrt{|\det\|g_{ij}\||}\; dx^1\wedge\cdots\wedge dx^n$

$\nabla g = 0 \Rightarrow \nabla \mathrm{Vol} = 0 \Rightarrow \mathrm{Hol}_x \cdot \mathrm{Vol}_x = \mathrm{Vol}_x$

$\Rightarrow \mathrm{Hol}_x \subset SO(n)$

$\Leftarrow$: $\mathrm{Hol}_x \subset SO(n)$ $\quad \exists\ \omega_x$ n-form

$\mathrm{Hol}_x \cdot \omega_x = \underline{\omega_x}$

$\Rightarrow \exists\ \omega \quad \nabla\omega = 0 \Rightarrow$

ω is non-vanishing

$\Rightarrow$ M is orientable. $\square$

$(M, g) \rightsquigarrow \nabla$

$x^1, \dots, x^n$

$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$

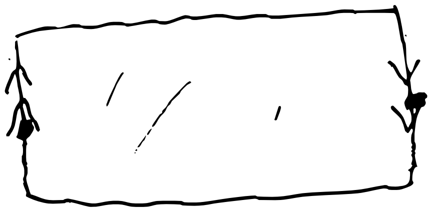
$g_{ij} := g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^{k} \frac{\partial}{\partial x^k}$

$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \dots\right)$

$\nabla_{X^i \frac{\partial}{\partial x^i}} Y^j \frac{\partial}{\partial x^j} = \dots$

Möbius



M

$\subset \mathbb{R}^2$ $\quad g = g_E$

g is def on M

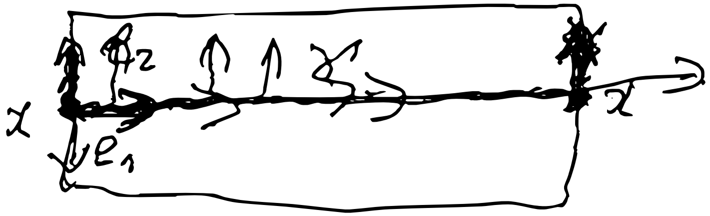
g is ~~fleat~~ flat $\quad R = 0$

$\mathrm{hol}_x(g) = 0$

$\Rightarrow \mathrm{Hol}^0_x(g) = \{ \mathrm{id} \}$

$$\pi_1(M, x) \to \mathrm{Hol}_x(g) / \mathrm{Hol}^0_x(g) =$$

$$= \mathrm{Hol}_x(g)$$



$\pi_1(M, x) = \mathbb{Z}$

$\tau_\gamma e_1 = e_1$

$\tau_\gamma e_2 = -e_2$

$\tau_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\tau_\gamma^2 = \mathrm{id}$ $\quad \Rightarrow \mathrm{Hol}_x(g) = \{ \mathrm{id}, \tau_\gamma \} = \mathbb{Z}_2$

$\tau_\gamma \notin SO(2)$

$(p, 0)$ $(0, q)$

vector is covariant

V vector space

$X \in V$ $(1, 0)$

$f: V \to W$

$f^*: W^* \to V^*$

covariant

V^* $e^1, \dots, e^n$

V $e_1, \dots, e_n$ | $f_1, \dots, f_n$

$X = X^i e_i$

$\theta = \theta_j e^j$

$\theta(X) = X^j \theta_j$

$\underline{e_{1'}, \dots, e_{n'}}$ $e^{1'}, \dots, e^{n'}$

$\underbrace{e_i = A_i^{i'} e_{i'}}$ $e_{i'} = A_{i'}^{i} e_i$

$A_i^{j'} A_{j'}^{k} = \delta_j^k$

$\underbrace{X^i = \boxed{A_{i'}^{i}} X^{i'}}$ $\underbrace{\theta_i = \boxed{A_i^{j'}} \theta_{j'}}$