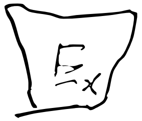
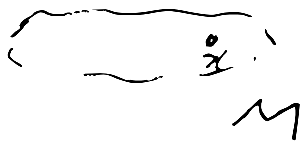
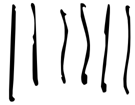
Lecture 3



$x \in M$ $\quad E_x \cong \mathbb{R}^m$





$$E = \bigcup_{x \in M} E_x$$

$$\begin{array}{c} M \\ \\ E \\ \downarrow p \\ M \end{array}$$

$\forall x \in M \quad p^{-1}(x) =: E_x$

$\forall x \in M \quad \exists \underset{\psi_x}{U} \subset M \qquad p^{-1}(U) \cong \underline{U \times} \mathbb{R}^m$

Example $\quad TM = \bigcup_{x \in M} T_x M$

$U \quad x^1 \dots x^n$

$X \in TM|_U$

$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$

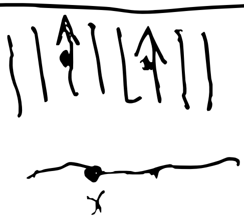
$X_x \in \underline{T_x} M$

$\dim TM = 2n \qquad \underline{x^1 \dots x^n}$

$\dim T_x M = n$

$X_x = \underline{X^i_x} \left(\frac{\partial}{\partial x^i} \right)_x \qquad \underline{X^1_x \dots X^n_x}$

<u>Section of E</u> $\quad s : M \to E \qquad p(s(x)) = x$



$s : x \mapsto s(x) \in E_x \qquad \Gamma(E)$

$\Gamma(TM) \Leftrightarrow$ vector fields on M

$\forall x \ \exists U \quad s_1, \dots, s_m \in \Gamma(E|_U) \quad \forall y \in U$

$(s_1)_y, \dots, (s_m)_y$ is a basis of E_y

Connection on E

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

$$\nabla_X s \qquad 1.\ 2.\ 3.$$

$\mathbb{R}$-bilinear

$$\nabla_{fX} s = f \nabla_X s$$

$$\nabla_X (fs) = (Xf) \cdot s + f \nabla_X s$$

locally: $U \quad \underline{x^1 \cdots x^n} \qquad \underline{s_1, \dots, s_m} \qquad i, j = 1, \dots, n$

$\alpha, \beta = 1, \dots, m$

$$\nabla_{\frac{\partial}{\partial x^i}} s_\alpha = \Gamma_{i\alpha}^{\beta} s_\beta$$

$\Gamma_{i\alpha}^{\beta}$ Christoffel symbols

$$X = X^i \frac{\partial}{\partial x^i} \qquad \xi = \xi^\alpha s_\alpha$$

$$\underbrace{\nabla_X s}, = X^i \nabla_{\frac{\partial}{\partial x^i}} \xi^\alpha s_\alpha = X^i \left(\left(\frac{\partial}{\partial x^i} \xi^\alpha \right) s_\alpha + \xi^\alpha \Gamma_{i\alpha}^{\beta} s_\beta \right)$$

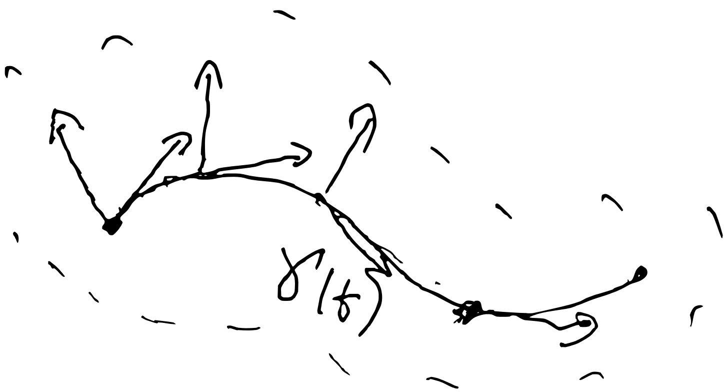
$$= \underbrace{X^i \left(\frac{\partial}{\partial x^i} \xi^\beta + \xi^\alpha \Gamma_{i\alpha}^{\beta} \right) s_\beta},$$

$s \in \Gamma(E)$ is $\parallel$ $\quad \nabla_X s = 0 \quad \forall X \in \Gamma(TM)$

$$\Leftrightarrow \underbrace{\frac{\partial \xi^\beta}{\partial x^i} + \Gamma^\beta_{i\alpha}\,\xi^\alpha = 0}_{\forall \beta, i} \qquad n \cdot m$$

$\mathbb{R}^n$

$\gamma(t)$ $\qquad \gamma : [a, b] \to M$



E, ∇

$$\boxed{s = \xi^\beta s_\beta}$$

$\dot\gamma(t) \in T_{\gamma(t)} M$

$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$

$$\dot\gamma(t) = \underline{\dot\gamma^i(t)} \left(\frac{\partial}{\partial x^i}\right)_{\gamma(t)}$$

$s(t) \in E_{\gamma(t)}$

$$\nabla_{\dot\gamma(t)} s(t) = \underline{\dot\gamma^i(t)} \left(\underline{\frac{\partial}{\partial x^i} \xi^\beta(t)} + \Gamma^\beta_{i\alpha}\,\xi^\alpha \right) s_\beta =$$

$$\xi^\beta(t) = \xi^\beta(\gamma^1(t), \dots, \gamma^n(t))$$

$$\frac{d\xi^\beta(t)}{dt} = \dot\gamma^i \frac{\partial \xi^\beta}{\partial x^i}$$

$$= \underline{\left(\frac{d\xi^\beta(t)}{dt} + \dot\gamma^i \Gamma^\beta_{i\alpha}\,\xi^\alpha \right) s_\beta}$$

S is called $\parallel$ <u>along</u> $\gamma(t)$

$$\nabla_{\dot\gamma(t)} S = 0$$

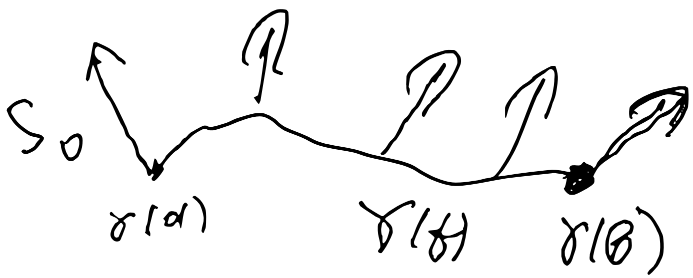
locally $\iff$ $\dfrac{d\xi^\beta(t)}{dt} + \dot\gamma^i(t)\,\Gamma_{i\,\alpha}^{\;\beta}\,\xi^\alpha = 0$

$\beta = 1,\dots,m$

$S \leftrightarrow \xi$

$$S(t) = \xi^\alpha(t)\, S_\alpha\big|_{\gamma(t)}$$

$\xi^1_0,\dots,\xi^m_0$ $\updownarrow$ $S_0 \in E_a$



$\exists!\ S(t)$ $\quad$ $S(a) = S_0$

$\tau_\gamma S_0 := S(b)$

$S_0 \in E_{\gamma(a)}$ $\quad$ $\tau_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$

Properties:

• τ_γ is linear $\quad \tau_\gamma(\xi_0 + \eta_0) = \xi(b) + \eta(b) =$

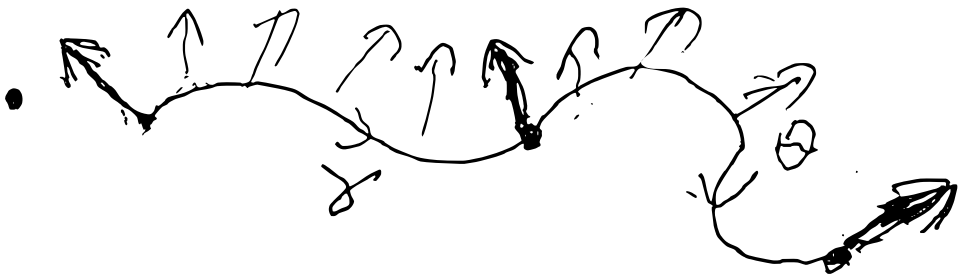
$\xi_0 + \eta_0 \quad\quad \xi_0 \rightsquigarrow \xi(t)$

$\eta_0 \rightsquigarrow \eta(t)$

$\underline{\xi(t) + \eta(t)} \quad\quad \xi_0 + \eta_0$

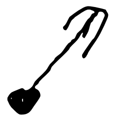
$= \tau_\gamma \xi_0 + \tau_\gamma \eta_0$

$\tau_\gamma(\lambda \xi_0) = \lambda \tau_\gamma(\xi)$

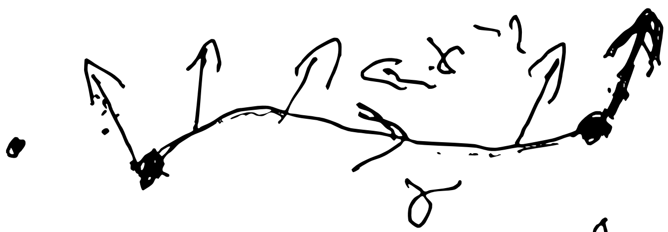
• 

$\underline{\tau_{\gamma * \theta}} = \underline{\tau_\theta \circ \tau_\gamma}$

• $\gamma(t) \equiv x \quad\quad \gamma = pt_x$



$\tau_\gamma = \underline{id_{T_x M}}$

• 

γ^{-1}

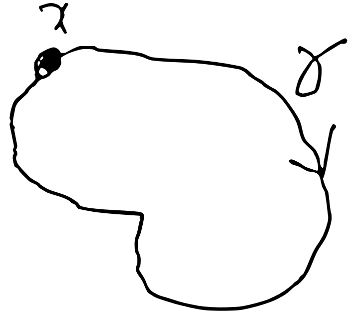
$\gamma : [a, b] \to M$

$\gamma^{-1} : [a, b] \to M$

$\gamma^{-1}(a) = \gamma(b)$

$\tau_{\gamma^{-1}} = (\tau_\gamma)^{-1}$

$$\mathrm{Hol}_x(\nabla) := \left\{ \tau_\gamma \;\middle|\; \right.$$

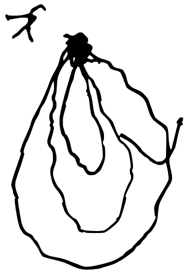


$$\left. \right\} \subset GL(E_x)$$

$\tau_\gamma : E_x \to E_x$ — $\tau_\gamma \in GL(E_x)$

$$\mathrm{Hol}^0_x(\nabla) := \left\{ \tau_\gamma \;\middle|\; \gamma \text{ is contractible} \right\}$$

restricted hol. group.



$\exists\, \mu_s \quad s \in [0,1]$

$\mu_1 = \gamma \qquad \mu_0 = \mathrm{pt}_x$

$$\mathrm{Hol}^0_x(\nabla) \subset \underline{\mathrm{Hol}_x(\nabla)} \subset GL(E_x)$$

$\underline{\mathrm{Hol}^0_x(\nabla)}$

$\tau_\gamma \in \mathrm{Hol}^0_x(\nabla) \qquad \exists\, \mu_s$

$\tau_{\mu_s} \in \mathrm{Hol}^0_x(\nabla) \qquad \tau_{\mu_1} = \tau_\gamma$

$\tau_{\mu_s} \Rightarrow \mathrm{Hol}^0_x(\nabla)$ is connected $\qquad \tau_{\mu_0} = \mathrm{id}_{T_xM}$

$\Rightarrow \mathrm{Hol}^0_x(\nabla)$ is a Lie subgr.

$$\pi_1(M,x) \to \underbrace{Hol_x(\nabla)/Hol_x^{0}(\nabla)}_{\text{discr}} \quad \text{surjective}$$

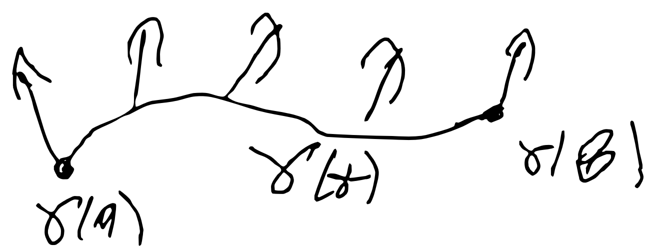
$Hol_x(\nabla)$

$$hol_x(\nabla) := \text{Lie alg of } Hol_x^{0}(\nabla), \quad Hol_x(\nabla)$$

$s \in \Gamma(E)$ is $\parallel$ $\iff$ $\nabla_X s = 0$ $\forall X \in \Gamma(TM)$

s is $\parallel$ along $\gamma(t)$ $\iff$ $\nabla_{\dot\gamma(t)} s = 0$

<u>Prop.</u> $\nabla s = 0 \iff \underline{\nabla_{\dot\gamma(t)} s = 0}$ $\forall \gamma(t)$



$$\underline{P_\gamma\, s(\gamma(a)) = s(\gamma(b))}$$

Theorem (fundamental principle for hol. gr.)

M is connected

$$\left\{ \xi \in \Gamma(E),\ \nabla \xi = 0 \right\} \Longleftrightarrow$$

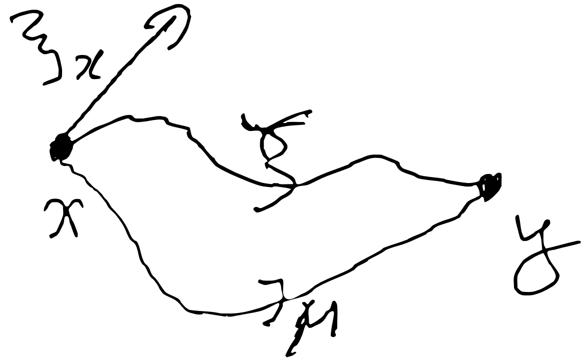
$$\Longleftrightarrow \left\{ \xi_x \in E_x \;\middle|\; a \xi_x = \xi_x \ \forall\, a \in Hol_x(\nabla) \right\}$$

Proof. $\Rightarrow$: $\nabla \xi = 0 \Rightarrow \xi$ is $\parallel$ along $\forall$ curve

$\Rightarrow \xi$ is $\parallel$ along all loops at x

$$\tau_\gamma \xi_x = \xi_x$$

$\Leftarrow$:



$$\xi_y := \tau_\gamma \xi_x$$

$$\tau_\gamma \xi_x \overset{?}{=} \tau_\mu \xi_x$$

$$\tau_{\gamma \circ \mu^{-1}} \xi_x = \underline{\tau_\mu^{-1} \tau_\gamma \xi_x} = \xi_x \ !$$

$\Rightarrow$ get $\xi \in \Gamma(E)$ $\qquad \nabla \xi = 0$ $\qquad \square$

E, ∇ suppose $\mathrm{Hol}(\nabla) = GL(E_x)$

$\Rightarrow$ the only $\parallel$ section is 0