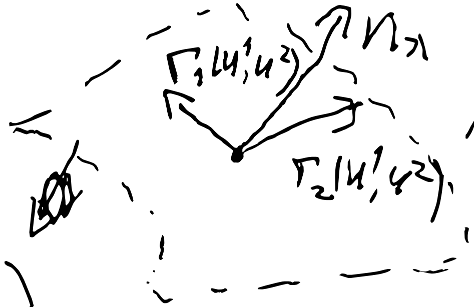
Lecture 2.

$r : U \to \mathbb{R}^3$, $\quad U \subset \mathbb{R}^2$ $\qquad u^1, u^2$



$M = r(U)$

$$r_i(u^1, u^2) = \frac{\partial r(u_0^1, u_0^2)}{\partial u^i} \qquad i = 1, 2$$

$$\in T_{r(u^1, u^2)} M$$

$r_1(u^1, u^2) \quad r_2(u^1, u^2) \qquad x = r(u^1, u^2) \quad T_x M$

$n_x \in T_x \mathbb{R}^3 \qquad n_x \perp r_{1x}, r_{2,x}$

$(n_x, n_x) = 1$

$r_{1x}, r_{2x}, n_x \quad T_x \mathbb{R}^3 = \mathbb{R}^3 \qquad x \in M$

n

$g_x = g_{\mathbb{R}^3, x}\big|_{T_x M \times T_x M} \qquad g_{x\,ij} := (r_{ix}, r_{jx})$

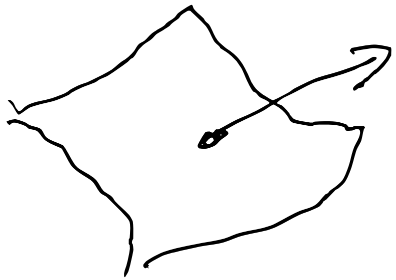
$X, Y \in T_x M \qquad X = X^1 r_{1x} + X^2 r_{2x} = X^i r_{ix}$

$Y = Y^i r_{ix} \qquad g_x(X, Y) = (X^i r_{ix}, Y^j r_{jx}) =$

$= X^i Y^j g_{x\,ij}$

$$r_{ij} := \frac{\partial r_j}{\partial u^i}(u^1,u^2) \in \mathbb{R}^3 \qquad r_{ij} = r_{ji}$$

$$T\mathbb{R}^3\big|_M$$



$$r_{ij}(u^1,u^2) = \Gamma^k_{ij}\, r_k(u^1,u^2) + b_{ij}\, n =$$

$$= \Gamma^1_{ij}\, r_1(u^1,u^2) + \Gamma^2_{ij}\, r_2(u^1,u^2) + b_{ij}\, n$$

PROP. $$\Gamma^k_{ij} = \frac{1}{2}\, g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right)$$

$$g_{ij} = (r_i, r_j) \qquad g^{kl} g_{lr} = \delta^k_r$$

Γ^k_{ij} depend only on g_{ij}!

PROOF OF PROP. $$\frac{\partial g_{ij}}{\partial u^k} = \frac{\partial (r_i, r_j)}{\partial u^k} = (r_{ik}, r_j) + (r_i, r_{jk})$$

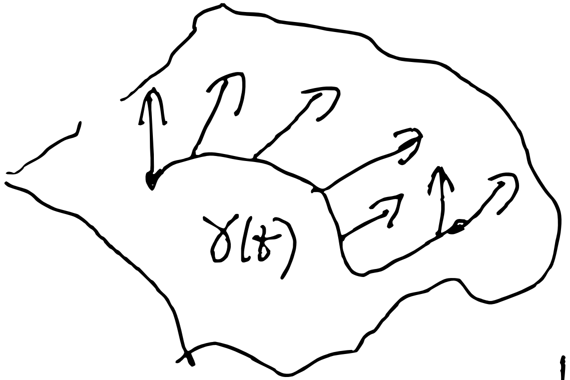
$$= (\Gamma^l_{ik}\, r_l, r_j) + \cdot = \Gamma^l_{ik}\, g_{lj} + \Gamma^l_{jk}\, g_{il}$$

$$\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{il}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l} = 2\Gamma^m_{ij} g_{ml}$$

$$g^{lk}$$

$$2\Gamma^m_{ij} g_{ml} \cdot g^{lk} = 2\Gamma^m_{ij} \delta^k_m =$$

$$= 2\Gamma^k_{ij} \qquad \square$$



$$V(t) \in T_{\gamma(t)} M$$

<u>Def</u> $V(t)$ is $\parallel$ along $\gamma(t)$

$$\dot V(t) := \frac{d V(t)}{dt} \in T_{\gamma(t)} \mathbb{R}^3$$

$$T_{\gamma(t)} \mathbb{R}^3 = T_{\gamma(t)} M \oplus \mathbb{R} n_{\gamma(t)}$$

$$V(t) \text{ is } \parallel \text{ along } \gamma(t) \overset{\text{def}}{\Longleftrightarrow}$$

$$\Longleftrightarrow \mathrm{Pr}_{T_{\gamma(t)} M} \dot V(t) = 0 \Longleftrightarrow \dot V(t) \in \mathbb{R} n_{\gamma(t)}$$

$$V(t) = V^1(t)\,\Gamma_1(\gamma(t)) + V^2(t)\,\Gamma_2(\gamma(t)) =$$

$$= V^i(t) \cdot \Gamma_i(\gamma(t))$$

$$\dot V(t) = \dot V^i(t) \cdot \Gamma_i(\gamma(t)) + V^i(t) \cdot \frac{d}{dt}\Gamma_i(\gamma(t)) =$$

$$\gamma(t) = \Gamma(u^1(t), u^2(t))$$

$$= \dot V^i(t)\,\Gamma_i(\gamma(t)) + V^i(t) \cdot \underbrace{\frac{\partial \Gamma_i}{\partial u^j}}_{\Gamma_{ij}} \cdot \dot u^j(t)$$

$$P_{T_{\gamma(t)}M}\, \dot V(t) = 0 \iff 0 = \left(\dot V^k(t)\,\Gamma_k + \Gamma_{ij}^k\,\Gamma_k\, V^i(t)\, \dot u^j(t) \right)$$

$$= \left(\dot V^k(t) + \Gamma_{ij}^k\, V^i(t)\, \dot u^j(t) \right) \Gamma_k$$

$$\iff \underbrace{\dot V^k(t) + \Gamma_{ij}^k\, V^i(t)\, \dot u^j(t) = 0}_{k=1,2}$$

$$u^j(t) = \gamma^j(t)$$

$$\dot u^j(t) = \dot \gamma^j(t)$$

parallel.

$\exists ! \ V(t) \in T_{\gamma(t)} M$

$V(0) = V_0$

$\gamma(t) \quad V(t)$ arbitrary v.f. along $\gamma(t)$

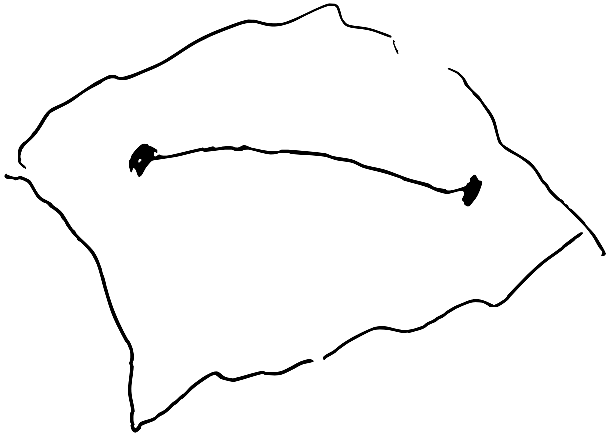
<u>Def.</u> $\nabla_{\dot\gamma(t)} V(t) := \mathrm{Pr}_{T_{\gamma(t)} M} \dot V(t)$

Cor. $V(t)$ is $\parallel$ along $\gamma(t)$ $\iff$ $\nabla_{\dot\gamma(t)} V(t) = 0$

$\gamma(t)$ is a geodesic ~~curve~~

if $\dot\gamma(t)$ is $\parallel$ along $\gamma(t)$

$\iff \nabla_{\dot\gamma(t)} \dot\gamma(t) = 0$



$$\Gamma^k_{ij}(\gamma(t))$$

$$\nabla_{\dot\gamma(t)} V(t) = \left(\dot V^k(t) + \Gamma^k_{ij} V^i(t)\, \dot u^j(t) \right) \Gamma_k$$

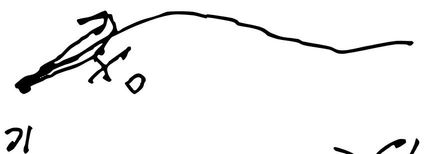
$$\dot u^j(t) = \dot\gamma^j(t)$$

$\gamma(t)$ geodesic

$$\nabla_{\dot\gamma} \dot\gamma(t) = 0$$

$$\Leftrightarrow \nabla_{\dot\gamma(t)} \qquad V^k = \dot\gamma^k$$

$$\Leftrightarrow \ddot\gamma^k + \Gamma^k_{ij} \cdot \dot\gamma^i \dot\gamma^j = 0$$



$\exists!\, \gamma(t)$

$\gamma(0) = x$

$\dot\gamma(0) = X_0$

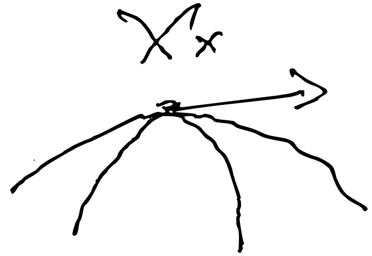
$\nabla_{\dot{\gamma}(t)} V(t) \in T_{\gamma(t)} M$

$X_x \in T_x M$

V v.f. def. in a neigh of x

$T_x M \ni \nabla_{X_x} V := \nabla_{\dot{\gamma}(t)} V(\gamma(t))\big|_{t=0}$

$\gamma(t)$ $\quad \gamma(0)=x \quad \dot{\gamma}(t) = X_x$



V v.f. on M

X v.f. on M

$x \in M \qquad X_x \in T_x M \qquad \nabla_{X_x} V \in T_x M$

$\nabla_X Y \in \Gamma(TM)$

$$\underline{\nabla} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

1. $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$

$f, g \in C^\infty M$

2. $\nabla_X (\alpha Y + \beta Z) = \alpha \nabla_X Y + \beta \nabla_X Z$

$\alpha, \beta \in \mathbb{R}$

3. $\nabla_X (fY) = (Xf) \cdot Y + f \nabla_X Y$

$$X = X^i \Gamma_i \qquad Y = Y^j \Gamma_j$$

$$\nabla_X Y = \nabla_{X^i \Gamma_i} Y^j \Gamma_j =$$

$$= X^i \underbrace{(\Gamma_i Y^j)}_{\frac{\partial Y^j}{\partial x^i}} \cdot \Gamma_j + X^i Y^j \underbrace{\nabla_{\Gamma_i} \Gamma_j}_{\Gamma_{ij}^k \Gamma_k}$$

$$\boxed{\Gamma_i = \frac{\partial}{\partial x^i}}$$