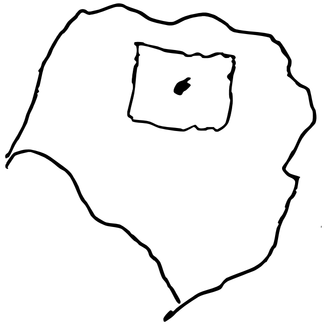
Holonomy groups of pseudo-Riemannian manifolds

Lecture 1.

<u>Motivation</u>

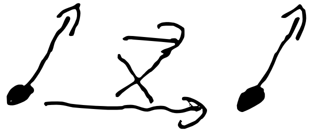
(M,g) Riemannian $\qquad x\in M$

$g_x : T_xM\times T_xM \to \mathbb{R}$



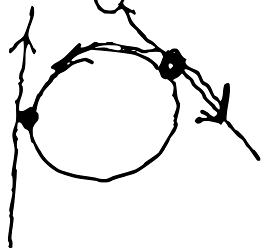
$\mathbb{R}^n$

$T_x\mathbb{R}^n = \mathbb{R}^n$



$g \rightsquigarrow$ „connection" $\nabla \rightsquigarrow$

$\rightsquigarrow \parallel$ transport



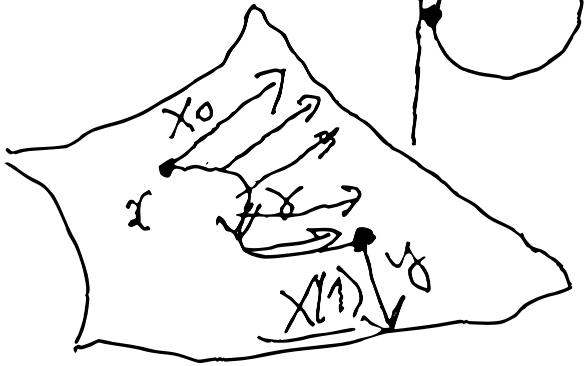
$\gamma : [0,1]\to M$

$\gamma(0)=x \qquad \gamma(1)=y$

$X_0\in T_{\gamma(0)}M$

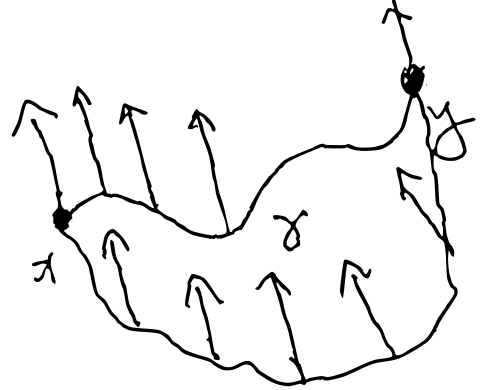
$\exists!\ X(t) \qquad X(0)=X_0$

$X(1)\in T_{\gamma(1)}M$



$\tau_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$ linear isomorphism

$\mathbb{R}^n$



$T_z\mathbb{R}^n = \mathbb{R}^n \quad \forall z \in \mathbb{R}$

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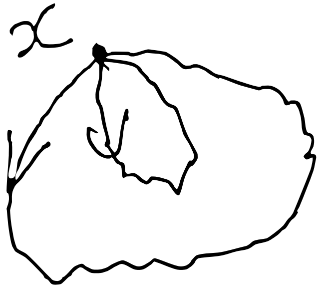
S^2

$\mathrm{Hol}_x(M,g) = \{ \tau_\gamma \mid \gamma \text{ is a loop at } x \}$



$\tau_\gamma : T_xM \to T_xM$

$\mathrm{Hol}_x(\mathbb{R}^n, g_{Euc}) = \{ id_{T_xM} \}$

$\mathrm{Hol}_x(S^2, g_{S^2}) = SO(2) = S^1$

M is connected $\mathrm{Hol}_x \cong \mathrm{Hol}_y$

$\mathrm{Hol}_y \to \mathrm{Hol}_x$

$\tau_\gamma \mapsto \tau_{\mu * \gamma * \mu^{-1}}$

$= \tau_{\mu^{-1}} \circ \tau_\gamma \circ \tau_\mu$

$\mathrm{Hol}_x = \tau_{\mu^{-1}} \circ \mathrm{Hol}_y \circ \tau_\mu$

$\forall \gamma, \mu \quad \tau_\gamma = \tau_\mu$

$\tau_{pt} = \mathrm{id}_{T_x M}$

$\Rightarrow \tau_\gamma = \mathrm{id}_{T_x M}$

$\underbrace{\mathrm{Hol}_x = \{\mathrm{id}_{T_x M}\}}_{\text{flat}} \Leftrightarrow (M, g)$ is locally isometric to $(\mathbb{R}^n, g_E)$

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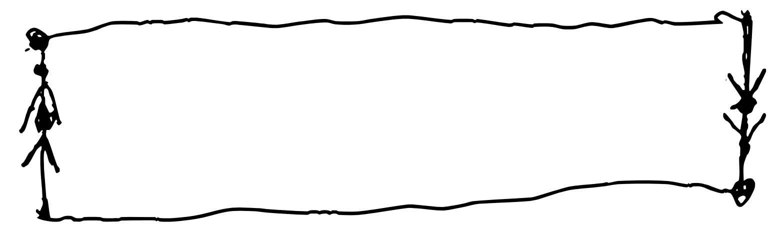
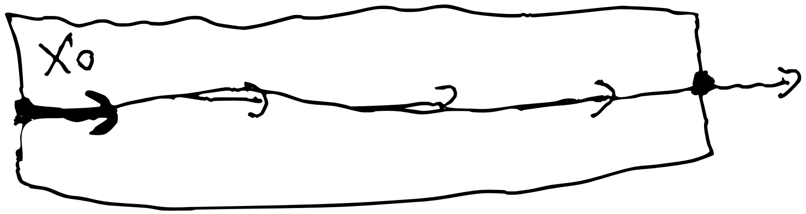
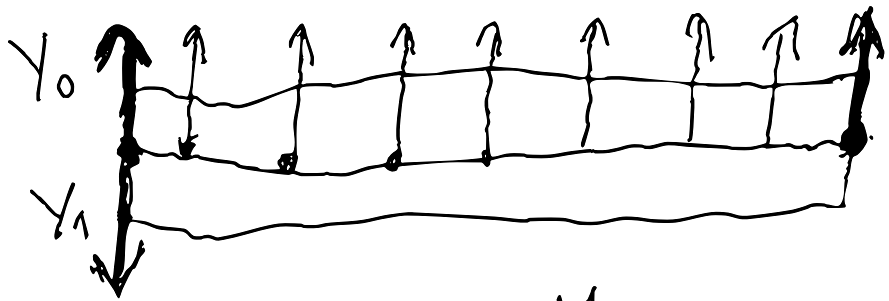
not flat $\Leftrightarrow$ „curved"

Möbius Band





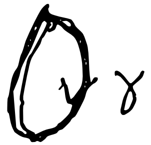
$\tau_\gamma X_0 = X_0$



$\tau_\gamma Y_0 = Y_1$

$\tau_M Y_0 = Y_0$

$\tau_M Y_0 = \tau_\gamma \cdot \tau_\gamma Y_0 = Y_0$



$M = \gamma * \gamma$

$\tau_M \neq 0$

$$\mathrm{Hol}(M, g) = \mathbb{Z}_2 = \{1, -1\} = $$

$$= \{\mathrm{id}_{T_x M}, \ -\mathrm{id}_{T_x M}\}$$

$1 \quad -1$

$\bullet \quad \bullet$ $\quad M$ is not orientable

$\mathbb{R}^n \quad \bullet \qquad S^2 \quad \cdot \ S^1$

Motivation

$r : U \to \mathbb{R}^3$ Euclidean space $(\cdot\,,\cdot)$

$U \subset \mathbb{R}^2$

u^1, u^2 on U

$r(u^1, u^2)$

$U \hookrightarrow r(U) = M$

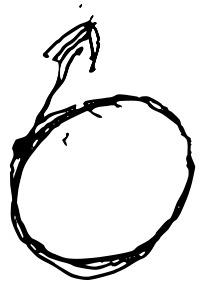
$$r(u^1,u^2) = (x^1(u^1,u^2),\ x^2(u^1,u^2),\ x^3(u^1,u^2))$$

$$g_x := g_{\mathbb{R}^3,x}\big|_{T_xM \times T_xM}$$

$$\underline{r_1}(u^1,u^2) = \frac{\partial r(u^1,u^2)}{\partial u^1} \qquad r_2(u^1,u^2) = \cdots$$

$$\gamma_1(t) = r(u^1+t, u^2)$$

$$T_{r(u^1,u^2)}M = \langle r_1(u^1,u^2), r_2(u^1,u^2)\rangle$$



$f : M \to \mathbb{R}$

$\Gamma_1(u^1, u^2) f = \frac{\partial f}{\partial u^1}(u^1, u^2$

$f(\Gamma(u^1, u^2))$

$\gamma_1(t) = (u^1 + t, u^2)$

$$\Gamma_1(u^1, u^2) f := \frac{d f(\Gamma(u^1 + t, u^2))}{dt}\Big|_{t=0} =$$

$$= \frac{\partial f}{\partial t}(u^1, u^2)$$

$f(u^1, u^2) =$

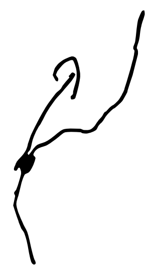
$= f(\Gamma(u^1, u^2))$

$$\Gamma_1(u^1, u^2) = \frac{\partial}{\partial u^1}(u^1, u^2) =$$

$$= \partial_{u^1}(u^1, u^2)$$

$X_x : C^\infty M \to \mathbb{R}$

$X_x(f \cdot g) = (X_x f) \cdot g(x) + f \cdot (X_x g)$



$\Rightarrow \exists\ \gamma(t) \quad \gamma(0) = x$

$\dot{\gamma}(t) = x_*$

$$X_x f = \frac{d}{dt} f(\gamma(t))\Big|_{t=0}$$