Observables and Factorization Algebras

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Outline

Definition

A prefactorization algebra F over M is a rule assigning a cochain complex $F(U)$ to each $U \in Ouv(M)$ together with

1. a cochain map $F(U)(V)$ for each $U \hookrightarrow V$
2. a cochain map $m^V_{U_1,\dots,U_n} : F(U_1) \otimes \cdots \otimes F(U_n)(V)$ for every finite collection of pairwise disjoint U_i
3. for $(V_j)_{j \in J}$ pairwise disjoint subsets of W and similarly $(U_{ij})_{i \in I_j}$ of each V_j, the following diagrams commutes

$$\begin{array}{ccc} \bigotimes\limits_{(i,j) \in \sqcup_k I_k} F(U_{ij}) & \xrightarrow{\quad m^W_{(U_{ij})_{\sqcup_k I_k}} \quad} & F(W) \\ & {\scriptstyle m^{V_j}_{(U_{ij})_{i \in I_j}}} \searrow \qquad \nearrow {\scriptstyle m^W_{(V_j)_{j \in J}}} & \\ & \bigotimes\limits_{j \in J} F(V_j) & \end{array}$$

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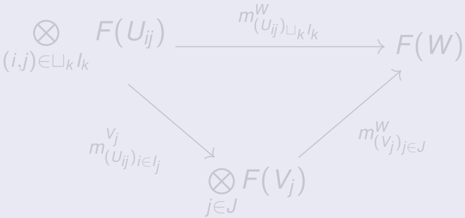

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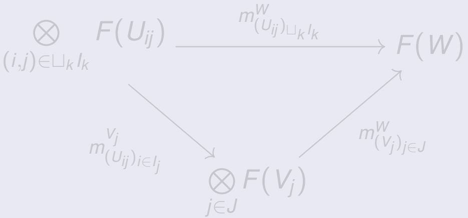

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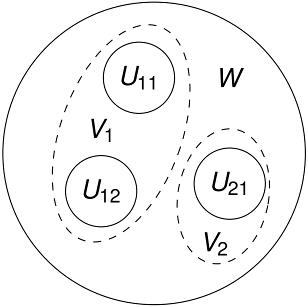
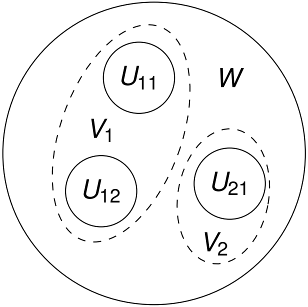


$$\begin{array}{ccc} F(U_{11}) \otimes F(U_{12}) \otimes F(U_{21}) & & \\ \downarrow & \searrow & \\ F(V_1) \otimes F(V_2) & \longrightarrow & F(W) \end{array}$$



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- algebra over little disk operad
- category $\mathrm{pFA}(M, C)$ - symmetric monoidal category $\mathrm{pFA}(M, \mathrm{Sym}\, C)$

Definition

A factorization algebra on M is a pFA on M such that for any $U \hookrightarrow M$ and Weiss cover $\{U_i\}_{i \in I}$

$$F(U) = \mathrm{colim}\left(\sqcup_{i,j} F(U_i \cap U_j) \rightrightarrows \sqcup_i F(U_i)\right)$$

- alternatively

$$\oplus_{i,j} F(U_i \cap U_j) \to \oplus_i F(U_i) \to F(U) \to 0$$

is exact $\Rightarrow$ homotopy FAs via Čech complexes

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Examples of FAs

- A_∞ structure via $F(U_1) \otimes \ldots \otimes F(U_n) \to F(\sqcup_i U_i)$
- if each $F(U) \to F(V)$ iso $\Rightarrow$ associative algebras
- let $P : \mathrm{Ouv} M^{op} \to C$, C abelian cat

$$F(U) \doteq \mathrm{Sym} P(U)$$

because structure maps induce

$$P(U_1) \oplus P(U_2) \to P(U_1 \sqcup U_2)$$

and

$$\mathrm{Sym}(P(U_1) \oplus P(U_2)) \cong \mathrm{Sym}(P(U_1)) \otimes \mathrm{Sym}(P(U_2))$$

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Classical observables

- let $\mathrm{Sol}(U)$ denote solutions to variational problem on $U \hookrightarrow M$ (classical field theory)
- define observables

$$\mathrm{Obs}(U) = \mathscr{O}(\mathrm{Sol}(U)) \tag{1}$$

- factorization algebra:

$$\mathrm{Sol}(U) \to \mathrm{Sol}(V_1) \times \ldots \times \mathrm{Sol}(V_n)$$

induces (via pullback)

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Quantization

- 2 approaches
- Costello:
  - work over $\mathbb{C}[[\hbar]]$, deform the differential:

$$\mathrm{d}(ab) = (\mathrm{d}a)b + (-)^{|a|}a(\mathrm{d}b) + \hbar\{a,b\}$$

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  - enforce QME; Wilsonian renormalization

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  - "keep" the differential, deform the product

$$e^{\hbar \partial_P}(A \star_P B) = (e^{\hbar \partial_P} A)(e^{\hbar \partial_P} B)$$

  - $\mathrm{d} = \{S, \_\} \to \mathrm{d}_\star = e^{-\hbar \partial_P} \circ \{S, \_\} \circ e^{\hbar \partial_P} \doteq \{S, \_\} + \hbar \Delta$
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- in general, gauge fixes can be organized simplicially $\Rightarrow$ QFTs form a derived stack

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  - functor $\mathscr{A} : \mathrm{Ouv}(M) \to \mathrm{Ass}_{\mathbb{K}}$
  - U, V causally disjoint: $\mathscr{A}(U)$, $\mathscr{A}(V)$ commute
  - time-slice axiom: $\mathscr{A}(U) \xrightarrow{\cong} \mathscr{A}(V)$ if $U \subseteq V$ contains Cauchy surface
- classically equivalent (just BV formalism)
- quantum mechanically: sometimes (Yau, Gwilliam-Rejzner)
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