Ricci solitons on k-symmetric Lorentzian manifolds

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A Lorentzian manifold $(\mathcal{M}, g)$ is a Ricci soliton iff there is a vector field S on $\mathcal{M}$ such that:

$$L_S g + r = \lambda g, \tag{1}$$

where r – Ricci curvature tensor, $\lambda \in \mathbb{R}$, $L_S g$ – Lie derivative of the metric tensor along S. Ricci solitons were pioneered by Hamilton[1].
A vector field S is a Killing field iff:

$$L_S g = 0.$$

[1] Hamilton R. S. The Ricci flow on surfaces // Contemporary Mathematics. 1988. V. 71

Let R be the Riemann curvature tensor of the metric g. If

$$\nabla^{k-1}R \neq 0, \nabla^k R = 0,$$

then $(\mathcal{M}, g)$ is called k-symmetric Lorentzian manifold. One-symmetric manifolds are the same as locally symmetric. When g is Riemannian metric, $\nabla^k R = 0$ for some $k > 1$ implies $\nabla R = 0$. 2-symmetric Lorentzian manifolds are described in articles[2] below[3].
A Lorentzian manifold $(\mathcal{M}, g)$ is called a pp-wave iff on $\mathcal{M}$ exists a vector field $V \neq 0$, such that:

$$g(V, V) = 0, \nabla V = 0, \mathrm{R}\,|_{V^\perp \wedge V^\perp} = 0.$$

[2] Alekseevsky D.V., Galaev A.S. Two-symmetric Lorentzian manifolds // Journal of Geometry and Physics. 2011. V. 61, N. 12 P. 2331-2340.

[3] Blanco O. F., Sánchez M., Senovilla J. M. Structure of second-order symmetric Lorentzian manifolds // Journal of the European Mathematical Society. 2013. V. 15, P. 595-634.

Local coordinates on pp-waves

Lemma

Let $(\mathcal{M}, g)$ be a pp-wave and let $p \in \mathcal{M}$. Then there a are local coordinates $\phi = (v, x = (x^1, \ldots, x^n), u)$ on a neighbourhood U of p and a function $h \in C^\infty(\phi(U))$, $h = h(u, x)$ not depending on v such that

$$g = 2du(dv + (h \circ \phi)du) + \delta_{ij} dx^i dx^j.$$

These coordinates are usually called Brinkmann coordinates[a]. Moreover, these coordinates can be chosen such that $h(u, 0) = 0$, $\frac{\partial h}{\partial x^i}(u, 0) = 0$ for all u in a neighbourhood of 0. Such coordinates are called normal Brinkmann coordinates centered at p.

[a] H. W. Brinkmann. Einstein spaces which are mapped conformally on each other. Math. Ann., 94(1):119–145, 1925.

Killing equation and Ricci solitons

NB. One can find general solution of the Ricci soliton equation using a particular solution and general solution of the Killing equation, therefore we need the following result[4]

Theorem (Globke W., Leistner T., see below)

Let $(\mathcal{M}^{n+2}, g)$ be locally indecomposable pp-wave. Then in normal Brinkmann coordinates $(v, x^1, \ldots, x^n, u)$ any Killing field has the following form:

$$K = (c - av - \dot{\Psi}x)\partial_v + (\Psi + Fx)^i\partial_i + (au + b)\partial_u, \tag{2}$$

where $a, b, c \in \mathbb{R}, F \in \mathfrak{so}(n)$, $\Psi : u \mapsto \Psi(u) \in \mathbb{R}^n$ is a solution of the equation:

$$\ddot{\Psi}^\top x - \mathrm{grad}(h)^\top(\Psi + Fx) - (au + b)\dot{h} - 2ah = 0. \tag{3}$$

[4] Globke W., Leistner T. Locally homogeneous pp-waves // Journal of Geometry and Physics. 2016. V. 108.

Theorem (A. S. Galaev, D. V. Alekseevsky)

Let $(\mathcal{M}, g)$ be a locally indecomposable Lorentzian manifold of dimension $n+2 \geq 4$. Then $(\mathcal{M}, g)$ is two-symmetric if and only if locally there exist coordinates $v, x^1, \ldots, x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + H(du)^2, H = (H_{ij}u + F_{ij})x^i x^j$$

where H_{ij} is a nonzero diagonal matrix, F_{ij} is a symmetric matrix.

Some important results

Theorem (EOR)

Let $(\mathcal{M}, g)$ be locally indecomposable 2-symmetric Lorentzian manifold of dimension $n \geq 4$. Then the Ricci soliton equation on $(\mathcal{M}, g)$ has a particular solution in special coordinates for any $\lambda \in \mathbb{R}$.

Scheme of the proof[5]:

$$\begin{cases} 2U_v = 0 \\ U_{x^j} + X_{j,v} = 0 \\ U_u + V_v = \lambda \\ 2X_{j,x^j} = \lambda \\ X_{j,x^i} + X_{i,x^j} = 0 \\ HU_{x^j} + X_{j,u} + V_{x^j} = 0 \\ -\frac{1}{2}\Delta H + 2U_u H + 2V_u + UH_u + \\ \quad + \sum_{j=1}^{n} X_j H_{x^j} = \lambda H \end{cases} \qquad (4)$$

[5] Ernst I.V., Oskorbin D.N., Rodionov E.D. Ricci Solitons on 3-Symmetric Lorentzian Manifolds // Izvestija of the ASU 2018. Issue 1, V. 99

Then the particular solution has the form:

$$
\begin{aligned}
S &= (V, \{X^j\}, U), \\
V &= \lambda v + \tfrac{u^2}{4} \sum H_{ii} + \tfrac{u}{2} \sum F_{ii}, \\
X^j &= \tfrac{\lambda x^j}{2}, \\
U &= 0.
\end{aligned}
$$

Let $(\mathcal{M}, g)$ be 2-symmetric Lorentzian manifold of dimension 4. The Ricci soliton equation on $\mathcal{M}$ has a particular solution[6] for any $\lambda \in \mathbb{R}$. We are going to describe general solution of the Ricci soliton equation on $\mathcal{M}$. Let $S = (V, X, Y, U)$ be a vector field on $\mathcal{M}$ in coordinates (v, x, y, u) from Theorem 3 (Galaev, Alekseevsky). Let's write down the Ricci soliton equation in coordinates (v, x, y, u).

[6] Onda K., Batat W. Ricci and Yamabe solitons on second-order symmetric, and plane wave 4-dimensional Lorentzian manifolds // Journal of Geometry. 2014, V. 105 Issue 3, P. 561-575.

$$\left\{\begin{array}{rcl} U_v & = & 0 \\ U_x + X_v & = & 0 \\ U_y + Y_v & = & 0 \\ U_u + V_v & = & \lambda \\ 2X_x & = & \lambda \\ Y_x + X_y & = & 0 \\ HU_x + X_u + V_x & = & 0 \\ 2Y_y & = & \lambda \\ HU_y + Y_u + V_y & = & 0 \\ -\frac{1}{2}(H_{xx} + H_{yy}) + 2U_u H + & & \\ +2V_u + X\varphi_x + Y\varphi_y + U\varphi_u & = & \lambda H \end{array}\right. \qquad (5)$$

Theorem

If both matrices H_{ij}, F_{ij} are scalar then general solution of the Ricci soliton equation on four-dimensional 2-symmetric Lorentzian manifold has the form:

$$S = \left(\lambda v - x\eta' - y\beta' + \gamma, \frac{\lambda x}{2} + fy + \eta, \frac{\lambda y}{2} - fx + \beta, 0\right)$$

where $\gamma(u) = \frac{(H_{11}+H_{22})u^2}{4} + \frac{(F_{11}+F_{22})u}{2} + \gamma_0$, $\gamma_0, f \in \mathbb{R}$, η and β are solutions of the system:

$$\begin{cases} \eta''(u) & = & 2(H_{11}u + F_{11})\eta(u) + 2F_{12}\beta(u) \\ \beta''(u) & = & 2(H_{22}u + F_{22})\beta(u) + 2F_{12}\eta(u) \end{cases}$$

If one of H, F is not scalar then, in addition $f = 0$.

In these cases the dimension of the Killing's vector fields space is 5 and 6 respectively.

We will use the following[7]

Theorem (A. S. Galaev)

Let $(\mathcal{M}, g)$ be a locally indecomposable Lorentzian manifold of dimension $n+2 \geq 4$. Then $(\mathcal{M}, g)$ is three-symmetric if and only if locally there exist coordinates $v, x^1, \ldots, x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + H(du)^2, H = (H_{2ij}u^2 + H_{1ij}u + H_{0ij})x^i x^j$$

where $H_{2ij}, H_{1ij}, H_{0ij}$ are symmetric real matrices, the matrix H_{2ij} is non-zero and can be assumed to be diagonal.

[7] Galaev A.S. Classification of third-order symmetric Lorentzian manifolds // Class. Quantum Grav. 2015, V. 32, P. 15

Some results

Theorem (EOR)

Let $(\mathcal{M}, g)$ be locally indecomposable 3-symmetric Lorentzian manifold of dimension $n \geq 4$. Then the Ricci soliton equation on $(\mathcal{M}, g)$ has a particular solution in special coordinates for any $\lambda \in \mathbb{R}$.

Similarly to 2-symmetric case, the particular solution has the form:

$$\begin{aligned} S &= (V, \{X^j\}, U), \\ V &= \lambda v + \frac{u^3}{6}\sum H_{2ii} + \frac{u^2}{4}\sum H_{1ii} + \frac{u}{2}\sum H_{0ii}, \\ X^j &= \frac{\lambda x^j}{2}, \\ U &= 0. \end{aligned}$$

3-symmetric case, dim M = 4

Let $(\mathcal{M}, g)$ be 3-symmetric Lorentzian manifold of dimension 4. Using the previous theorem we get a particular soliton for the Ricci soliton equation:

$$\begin{aligned} S = (V, X = X^1, Y = X^2, U), \\ V = \lambda v + \frac{u^3}{6} \sum H_{2ii} + \frac{u^2}{4} \sum H_{1ii} + \frac{u}{2} \sum H_{0ii}, \\ X_j = \frac{\lambda x^j}{2}, \\ U = 0. \end{aligned}$$

Adding this to general solution of the Killing equation we get general solution of the Ricci soliton equation.

General solution

Theorem

If the matrices $H_{2ij}, H_{1ij}, H_{0ij}$ are scalar then general solution of the Ricci soliton equation on four-dimensional 3-symmetric Lorentzian manifold have the following form:

$$S = \left(\lambda v - x\eta' - y\beta' + \gamma, \frac{\lambda x}{2} + fy + \eta, \frac{\lambda y}{2} - fx + \beta, 0\right)$$

where $\gamma(u) = \frac{u^3}{6}\sum H_{2ii} + \frac{u^2}{4}\sum H_{1ii} + \frac{u}{2}\sum H_{0ii} + \gamma_0$, $\gamma_0, f \in \mathbb{R}$, η and β are solutions of the system:

$$\begin{cases} \eta''(u) &= 2(H_{211}u^2 + H_{111}u + H_{011})\eta(u) + 2(H_{112}u + H_{012})\beta(u) \\ \beta''(u) &= 2(H_{222}u^2 + H_{122}u + H_{022})\beta(u) + 2(H_{112}u + H_{012})\eta(u) \end{cases}$$

If one of $H_{2ij}, H_{1ij}, H_{0ij}$ is not scalar then, in addition $f = 0$.

Thank you for your attention!