Introduction to Computational Topology

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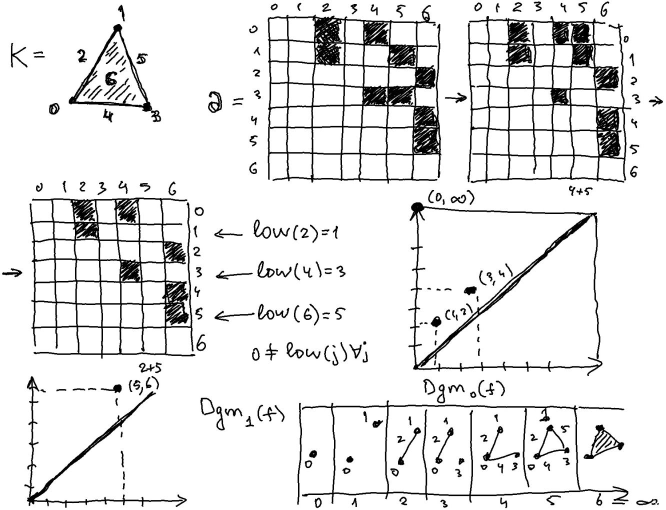
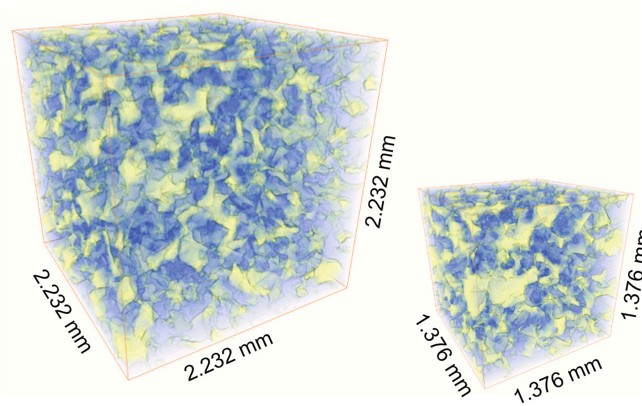
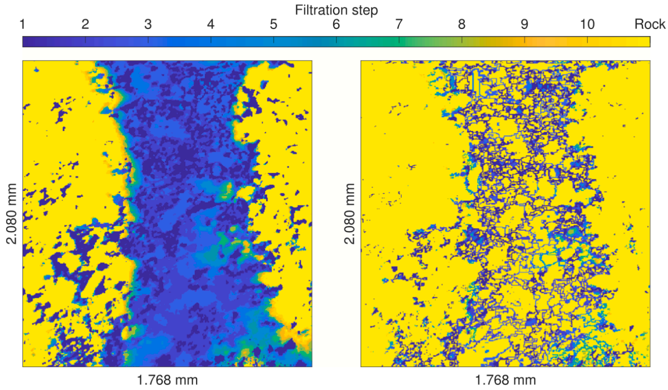
Example.



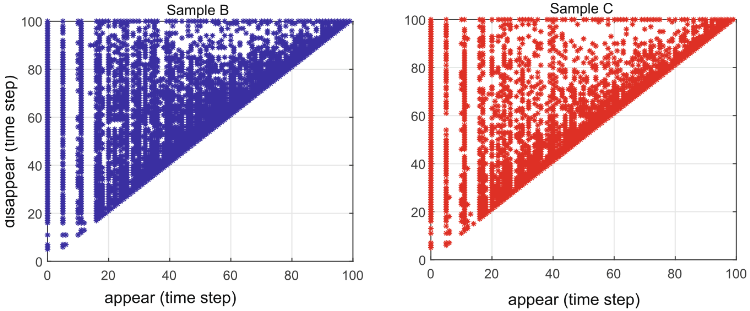
Two samples of sandstone



Chemical dissolution of sandstone gives topological filtration.



One-dimensional persistence diagrams (Dgm_1) of two samples under chemical dissolution of skeleton [Dmitriy Prokhorov, Vadim Lisitsa, Yaroslav Bazaikin. Digital image reduction for the analysis of topological changes in the pore space of rock matrix // Computers and Geotechnics. 2021. V. 136, 104171].



Stability of persistence diagrams

In what extent will $Dgm_p(f)$ change, if we change function f? We can try at first represent change of f as sequence of small changes.

Let $f : K \to \mathbb{R}$ and $g :\to \mathbb{R}$ be two monotonic functions. Define straight-line homotopy $F : K \times [0,1] \to \mathbb{R}$ between f and g by

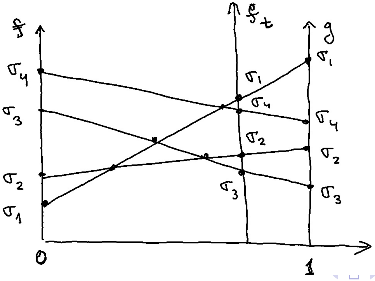
$$F(\sigma, t) = (1-t)f(\sigma) + tg(\sigma), \sigma \in K.$$

Let denote $f_t(\sigma) = F(\sigma, t)$, then $f_0 = f$, $f_1 = g$. Let σ is a face of τ. Then $f(\sigma) \leq f(\tau)$, $g(\sigma) \leq g(\tau)$ and therefore $f_t(\sigma) \leq f_t(\tau)$.

Lemma. *If f and g are monotonic then f_t is monotonic for all $t \in [0,1]$*

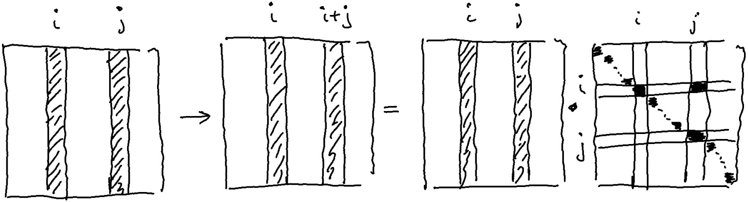
Suppose that f and g are injective, that is for all simplexes $\sigma, \tau \in K$ we have $f(\sigma) \neq f(\tau)$ and $g(\sigma) \neq g(\tau)$. Then f_t is injective with the exception of some particular t, where two simplexes can have the same value.

If we consider ordering of K, compatible with f then there will be finite number of changing this ordering keeping compatibility with f_t, and every change is transposing two simplexes.



How transposing two simplexes affects reducing procedure of boundary matrix ∂? And how this affects pairing $i = low(j)$?

Let ∂ is boundary matrix for some compatible ordering $\{\sigma_1, \ldots, \sigma_m\}$. We can represent matrix reducing as right multiplication:

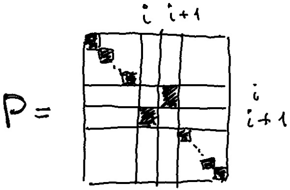


So we have $R = \partial V$ and if we take right inverse to V then

$$\partial = \partial VU = RU.$$

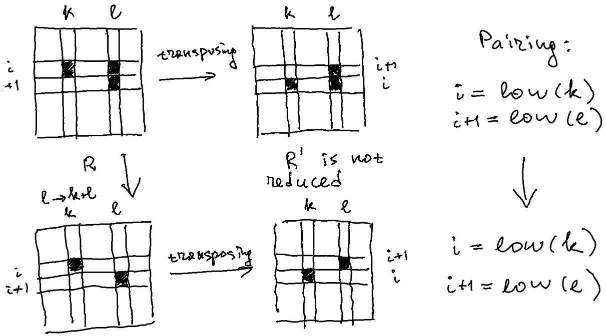
Recall that in this RU decomposition of boundary matrix ∂, R is reduced upper triangular and U is upper triangular and invertible.

Consider transposition of two simplexes σ_i and σ_{i+1}. Then new boundary matrix ∂' has the form: $\partial' = P\partial P$, $P^2 = E$.

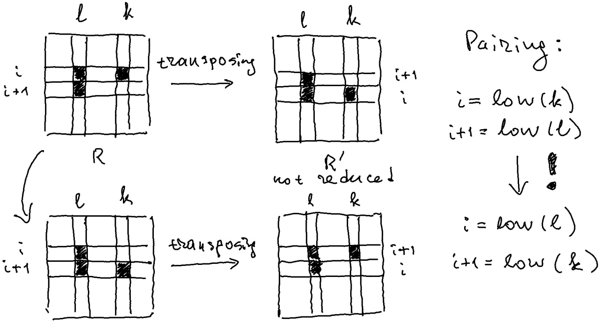


$$\partial' = P\partial P = PRUP = (PRP)(PUP) = R'U'$$

We need additionally correct R and U to provide $R'U'$ to be a correct RU decomposition.



Second possibility (remark change of pairing!):



Decomposition Lemma *Transposition of two simplexes* σ_i, σ_{i+1} *can change pairing* $i = low(k)$ *and* $i + 1 = low(l)$ *only in case* $\dim \sigma_i = \dim \sigma_{i+1}$ *and pairing after changing looks like* $i = low(l)$, $i + 1 = low(k)$.

How to measure the distance between two persistence diagrams X, Y? Let $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in Y$.

$$\|x - y\|_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Let $\eta : X \to Y$ be some bijection (it always exists because X and Y contain infinite number of points!). Let

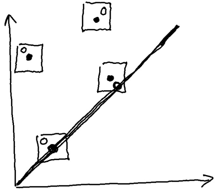
$$|\eta|_\infty = \sup_{x \in X} \|x - \eta(x)\|_\infty$$

We define *bottleneck distance* between persistence diagrams:

$$W_\infty(X, Y) = \inf_{\eta : X \to Y} |\eta|$$

(inf is considered over all bijection η).

If $W_\infty(X,Y) = a$ then every point of X belongs to square with edge length $2a$ with a center in some point of Y.



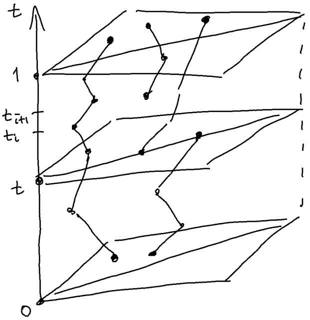
Properties of W_∞:

$$1) W_\infty(X,Y) = 0 \Leftrightarrow X = Y,$$

$$2) W_\infty(X,Y) = W_\infty(Y,X),$$

$$3) W_\infty(X,Z) \leq W_\infty(X,Y) + W_\infty(Y,Z).$$

Let $f, g : K \to \mathbb{R}$ be two monotonic functions and define homotopy $f_t = (1-t)f + tf,\ t \in [0,1]$. We obtain a family of persistence diagrams in $\bar{\mathbb{R}}^2 \rtimes [0,1]$, every point non-diagonal point has the form $x(t) = (f_t(\sigma), f_t(\tau), t)$, $\sigma, \tau \in K$ (adding σ gives new cycle, adding τ kills this cycle).



We can find finite set of values $0 = t_0. < t_1 < \ldots < t_{n+1}$ where pairing changes. This implies that on the interval $(t_i, t_{i+1}$ we have the same pairing and $x(t)$ is a line segment which connects two points on planes $t = t_i$ and $t = t_{i+1}$.

In the moment t_{i+1} (if the point is off diagonal) we have only two possibilities: or continue previous segment in the same direction; or pairing changes and we continue with another segment.

Decompose all off-diagonal points $x(t)$ into finite number of piecewise straight lines of the following three types:

1) starting at off-diagonal point of $Dgm_p(f_0)$ and finishing at off-diagonal point of $Dgm_p(f_1)$;

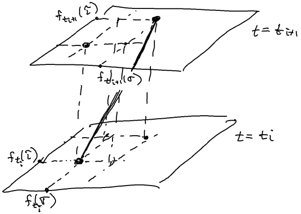
2) starting at off-diagonal point of $Dgm_p(f_0)$ and finishing on diagonal for some t;

3) starting on diagonal for some t and finishing at off-diagonal point of $Dgm_p(f_1)$;

We can continue diagonal points at $t \in (0, 1)$ by constant vertical lines and obtain one-to one correspondence (bijection) η between $Dgm_p(f)$ and $Dgm_p(g)$.

The lines we obtain is called *vines* and collection of all vines is *vineyard*.

Let estimate $|\eta|$ we construct.



The distance between endpoints of segment $[x(t_i)x(t_{i+1})]$ equals:

$$\|(f_{t_{i+1}}(\sigma), f_{t_{i+1}}(\tau)) - (f_{t_i}(\sigma), f_{t_i}(\tau))\|_\infty =$$

$$(t_{i+1} - t_i)\max\{|f(\sigma) - g(\sigma)|, |f(\tau) - g(\tau)|\}$$

Let $\nu \in K$ is a simplex with maximal value of $|f(\nu) - g(\nu)|$ which we can introduce as L_∞ distance on the space of functions on K:

$$\|f - g\|_\infty = \max_{\sigma \in K} |f(\sigma) - g(\sigma)| = |f(\nu) - g(\nu)|.$$

Then

$$\|(f_{t_{i+1}}(\sigma), f_{t_{i+1}}(\tau)) - (f_{t_i}(\sigma), f_{t_i}(\tau))\|_\infty \leq (t_{i+1} - t_i)|f(\nu) - g(\nu)|.$$

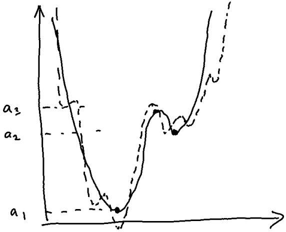
Then distance between endpoints of vineyard can be estimated by

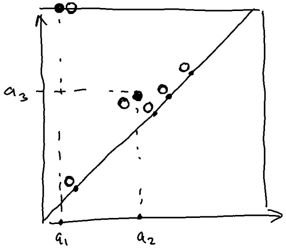
$$\|(f_0(\sigma), f_0(\tau)) - (f_1(\sigma), f_1(\tau))\|_\infty \leq |f(\nu) - g(\nu)| = \|f - g\|_\infty.$$

Stability Theorem. *Let K be simplicial complex and $f, g : K \to \mathbb{R}$ two monotonic functions. For each dimensions following inequality holds:*

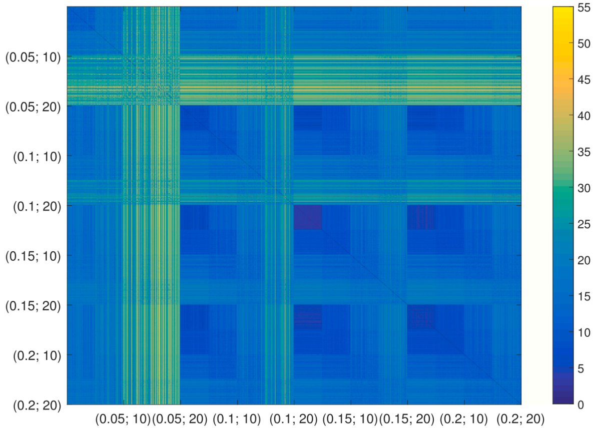
$$W_\infty(Dgm_p(f), Dgm_p(g)) \leq \|f - g\|_\infty.$$

Illustration in more general situation:





Bottleneck distance between barcodes corresponding different parameters of chemical dissolution of rock.



References.

[1] Herbert Edelsbrunner, John L. Harer. Computational Topology. An Introduction. AMS. 2010.

[2] Afra J Zomorodian. Topology for computing. Cambridge Univ Psess. 2005

Thank you for attention!