Introduction to Computational Topology

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Basic Concepts of Algebraic Topology

Given set X a topology τ on X is some family of subsets of X with the properties:

1) $\emptyset, X \in \tau$;

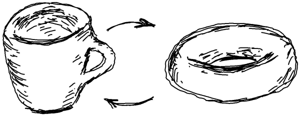
2) if $U_1, \ldots, U_n \in \tau$ then $\cap_{i=1}^{n} U_i \in \tau$;

3) if $U_\alpha \in \tau, \alpha \in A$ then $\cup_{\alpha \in A} U_\alpha \in \tau$.

Pair (X, τ) is called *topological space*.

This definition is the formal mathematical base for operation with intuitive concept of "being in neighborhood": we call subset $V \subset X$ to be a neighborhood of a point $p \in X$ if there exists $U \in \tau$ such that $p \in U \subset V$.

Let X, Y be two topological spaces. We say that map $f : X \to Y$ is continuous if $V \in \tau(Y)$ implies $f^{-1}(V) \in \tau(X)$. Bijective $f : X \to Y$ with both continuous f and f^{-1} is called *homeomorphism*.

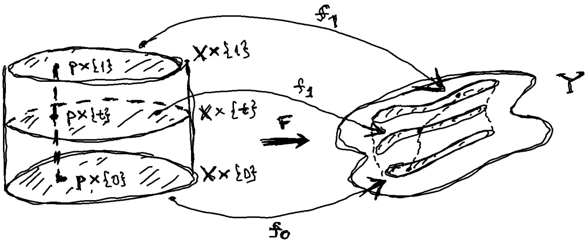


One of general problems of algebraic topology is to classify all topological spaces with respect to property to be homeomorphic (this problem is impossible to solve in general). Consider weaker problem of homotopy classification.

Let $f_0, f_1 : X \to Y$ be two continuous maps. *Homotopy* between f_0 and f_1 is a contnuous map $F : X \times [0,1] \to Y$ with the following properties:

$$F(p,0) = f_0(p), F(p,1) = f_1(p), p \in X.$$

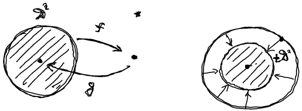
Denoting $f_t(p) = F(p,t)$ we can think of homotopy as a continuous family maps f_t connecting f_0 and f_1.



We say f and g are homotopic if there exists homotopy between them, write $f \sim g$. Map $f : X \to Y$ is called homotopy equivalence if there exists map $g : Y \to X$ such that:

$$f \circ g \sim id_Y, g \circ f \sim id_X.$$

If homotopy equivalence exists we say that X and Y are homotopy equivalent. Remark that homotopy equivalence doesn't have to be a bijection:



It is obvious that homeomorphism is homotopy equivalence so homotopy classification is weaker problem than homeomorphism classification.

Let we associate each topological space X with a certain group $H(X)$ satisfying the following properties:

1) every map $f : X \to Y$ is associated with a certain group homomorphism $f_* : H(X) \to H(Y)$;

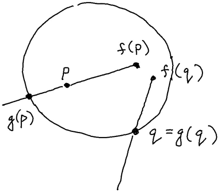
2) $id_* = id$, that is identical map $id(p) = p$ is associated with identical group homomorphism;

3) $(f \circ g)_* = f_* \circ g_*$;

4) if $f \sim g$ then $f_* = g_*$.

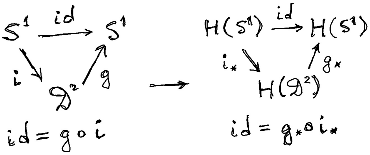
It is easy to derive from above properties that if X is homotopy equivalent to Y then $H(X)$ is isomorphic (as a group) to $H(Y)$. This allow to move the problem to group theory which can be much more constructive and easy to solve.

Brouwer's Fixed Point Theorem. Let $f : D^2 \to D^2$ is continuous. Then there exist fixed point $p \in D^2$: $f(p) = p$.



Idea of proof: assuming no fixed points we construct continuous map $g : D^2 \to S^1$ satisfies the property: $g(p) = p$, for all $p \in S^1$.

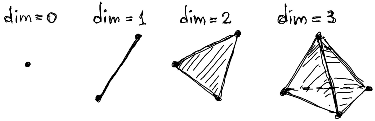
The latter property can be formulated on the language of commutative diagrams:



If now we construct some invariant H such that $H(D^2) = 0$ and $H(S^1) \neq 0$ then the contradiction follows from the right diagram (which finishes proof of the fixed point theorem).

What would be a constructive way to describe topological space? One approach is very successful both in algebraic and computational topology: simplicial complexes.

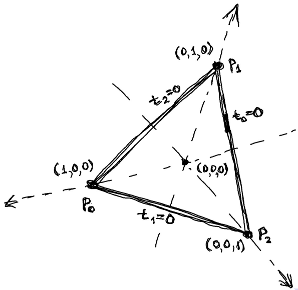
Elementary brick of simplicial complex: simplex of dimension n is defined as convex hull of $n + 1$ points in general position.



Let $p_0, \ldots, p_n \in \mathbb{R}^N$ are in general position for sufficiently large N. We denote by $\sigma = [p_0, \ldots, p_n]$ simplex generated by these points. If $p \in \sigma$ then

$$p = t_0 p_0 + t_1 p_1 + \ldots + t_n p_n, \sum_{i=0}^{n} t_i = 1,$$

where $(t_0, \ldots, t_n)$ are barycentric coordinates of p in σ.



Every subset $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$, $0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n$ defines face of σ of co-dimension k by equation $t_{i_1} = t_{i_2} = \ldots = t_{i_k} = 0$. Remaining coordinates defines barycentric coordinate system in the face.

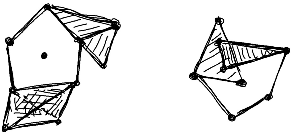
Points p_i, $i = 0, \ldots, n$ are faces of co-dimension n and is called vertexes of σ.

We denote $\partial_i \sigma$, $i = 0, \ldots, n$ the faces of co-dimension 1 (hyper-faces).

A simplicial complex K is a set of simplices that lies in $\mathbb{R}^N$ and satisfies the following conditions:

1) every face of a simplex from K is also in K;

2) the non-empty intersection of any two simplices from K is a face of both simplices.



(left: simplicial complex; right: not a simplicial complex)

Support of K is topological space $|K| = \cup K \subset \mathbb{R}^N$.

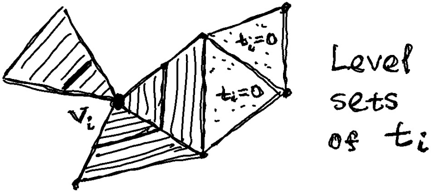
To storage simplicial complex we don't need to know all vertexes of all simplices. The concept of abstract simplicial complex reduces topological space to finite combinatorial data.

Let S be non-empty finite set and K be a collection of non-empty subsets of S. We say that K is *abstract simplicial complex* if for every set $\sigma \in K$ and for every $\tau \subset \sigma$, we have $\tau \in K$.

It is easy to see that putting S to be a set of all vertexes and K to consist of collection of vertexes of simplices we obtain abstract simplicial complex defined by simmplicial complex. Conversely, we always can restore simlicial complex by abstract simplicial complex.

We can expand barycentric coordinates to whole simlicial complex. For simlicial complex K let $V(K)$ denote the set of all vertexes of K, $V(K) = \{v_1, v_2, \ldots, v_n\}$.

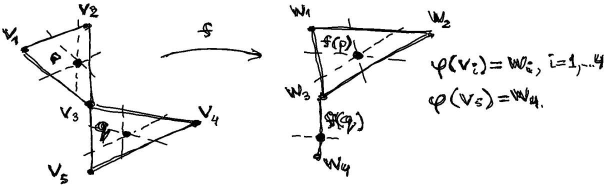
For every vertex v_i and simplex $\sigma \ni v_i$ we have barycentric coordinate $t_i : \sigma \to \mathbb{R}$. We can use the same notation for these coordinates in different simplexes because they are compatible in common faces. Put $t_i = 0$ for all points in all simplexes which doesn't contain v_i.



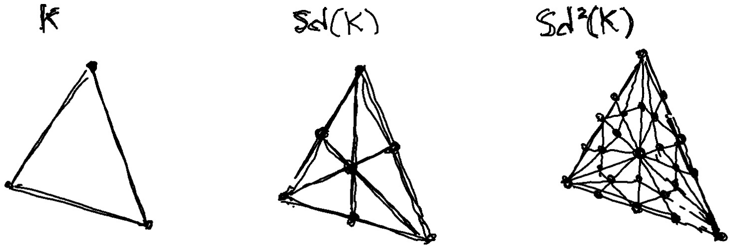
Let K and L be two simplicial complexes. Let $V(K) = \{v_1, \dots, v_n\}$ $V(L) = \{w_1, \dots, w_k\}$. Map $\varphi : V(K) \to V(L)$ is vertex map if for any simplex σ from K all vertexes of σ are mapped to vertexes of the unique simplex from L. This property correctly defines map $f : K \to L$ which we call *simplicial map*.

Now we can define continuous map $f : |K| \to |L|$ as follows:

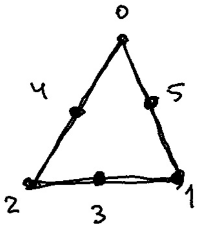
$$f(p) = \sum_{i=0}^{n} t_i(p)\varphi(v_i), p \in |K|.$$

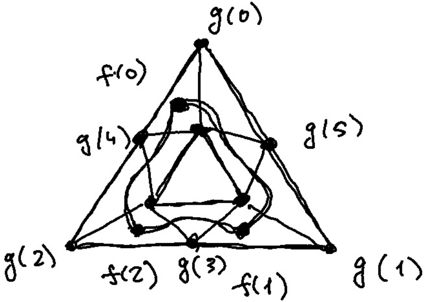


A simplicial complex L is a subdivision of another simplicial complex K if $|L| = |K|$ and every simplex in L is contained in a simplex in K. Consider one possibility of subdivision, barycentric subdivision $L = Sd(K)$:



Theorem. *Let K, L be simlicial complexes and $f : |K| \to |L|$ be continuous map (not simplicial!). Then there exist n-step subdivision $Sd^n(K)$ and simplicial map $g : Sd^n(K) \to L$ such that $g : |Sd^n(K)| = |K| \to |L|$ is homotopic to f.*



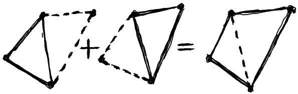


Let K be simplicial complex. For every integer $p \geq 0$ define group $C_p = C_p(K)$ of p-dimensional chains in K:

$$C_p = \left\{ \sum_j a_j \sigma_j^p \mid \dim(\sigma_j^p) = p, a_j \in \mathbb{Z}_2 = \{0, 1\} \right\}.$$

We can think of C_p as a vector space generated by basis consisting of all p-dimensional simplices with coefficients 0 or 1 with coordinate wise addition operation.

Coefficients satisfy relations: $0 + 0 = 1 + 1 = 0, 0 + 1 = 1 + 0 = 1$.



Define *boundary homomorphism* $\partial_p = \partial : C_p \to C_{p-1}$ by the following:

$$\partial(\sigma) = \sum_{i=0}^{p} \partial_i \sigma, \sigma \in K, \dim(\sigma) = p,$$

$$\partial(\sum_j a_j \sigma_j^p) = \sum_j a_j \partial(\sigma_j^p).$$

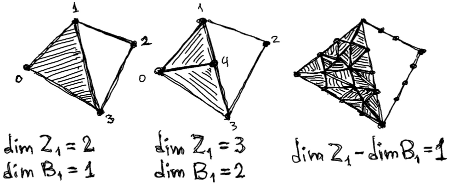
Theorem. $\partial^2 = \partial \circ \partial = 0.$

Proof:

A chain $c \in C_p$ is called *cycle* if $\partial(c) = 0$. We denote Z_p the group of all cycles of dimension p.

A chain $c \in C_p$ is called *boundary* if $c = \partial(c')$ for some $c' \in C_{p+1}$. The group of all boundaries is denoted by B_p.

We have $B_p \subset Z_p$ from above theorem.



Roughly speaking difference $\dim(Z_p) - \dim(B_p)$ measures number of p-dimensional "holes".

Define *n*-dimensional homology group $H_p(K)$ as follows:

$$H_p(K) = \frac{Z_p(K)}{B_p(K)} \; (factor - group \; of \; abelian \; groups)$$

Rank of homology group
$b_p(K) = rank(H_p(K)) = rank(Z_p(K)) - rank(B_p(K))$ is called
p-dimensional Betti number ("number of *p*-dimensional halls").

$$H_p = \underbrace{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2}_{b_p \text{ times}}$$

Let $f : K \to L$ be simplicial map. Then f induces hmomorphism

$$f : C_p(K) \to C_p(L)$$

with the property

$$\partial_p \circ f = f \circ \partial_p.$$

Then $f(Z_p(K)) \subset Z_p(L)$ and $f(B_p(K)) \subset B_p(L)$. This defines homomorphism

$$f_p : H_p(K) \to H_p(L).$$

Theorem. *If $f : |K| \to |L|$ is homotopic to $g : |K| \to |L|$ then $f_p = g_p$ for all p.*

We can expand definition of f_p to all continuous $f : |K| \to |L|$ applying simplicial map $g : Sd^n(K) \to L$.

Theorem. *Let K and L be two simplicial complexes and $|K|$ is homeomorphic to $|L|$. Then group $H_n(K)$ is isomorphic to $H_n(L)$. In particular, $b_n(K) = b_n(L)$ for all integer $n \geq 0$.*

Let prove one important consequence of this theorem.

Euler characteristics $\chi(K)$ of simplicial complex K is the number

$$\chi(K) = a_0 - a_1 + a_2 - \ldots + (-1)^k a_k + \ldots,$$

where $a_k = rank(C_k)$ be the number of k-dimensional simplices in K.

Euler-Poincaré Theorem. *For simplicial complex K we have*

$$\chi(K) = b_0(K) - b_1(K) + b_2(K) - \ldots + (-1)^k b_k(K) + \ldots,$$

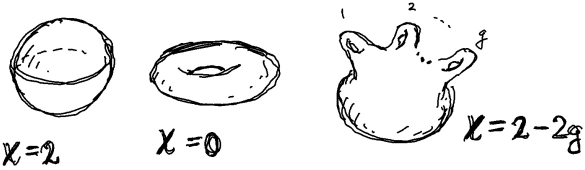
and $\chi(K) = \chi(L)$ in the conditions of the above theorem.

Proof.

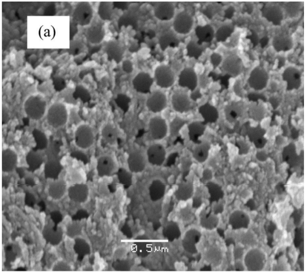
$$Z_i = B_i \oplus \frac{Z_i}{B_i} = B_i \oplus H_i,$$
$$C_i = Z_i \oplus \frac{C_i}{Z_i} = Z_i \oplus B_{i-1} = H_i \oplus B_i \oplus B_{i-1}.$$

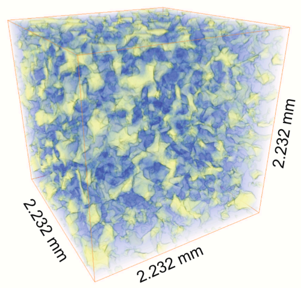
Then $a_i = rank(C_i) = b_i + rank(B_i) + rank(B_{i-1})$.

Now it is evident that
$\chi(K) = a_0 - a_1 + a_2 - a_3 + \ldots = b_0(K) - b_1(K) + b_2(K) - b_3(K) + \ldots$.



What are Euler characteristics and Betti numbers of the following topological spaces?





(left: SEM of aluminium oxide catalyst with artificially created macro-pores; right: the model of Bentheimer sandstone)

Thank you for attention!