Heterogeneous neurogeometry

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Joint work with Giovanna Citti

Conference in honor of Dmitri Alekseevsky
7-9 September 2020


SMI
Scuola Matematica Interuniversitaria
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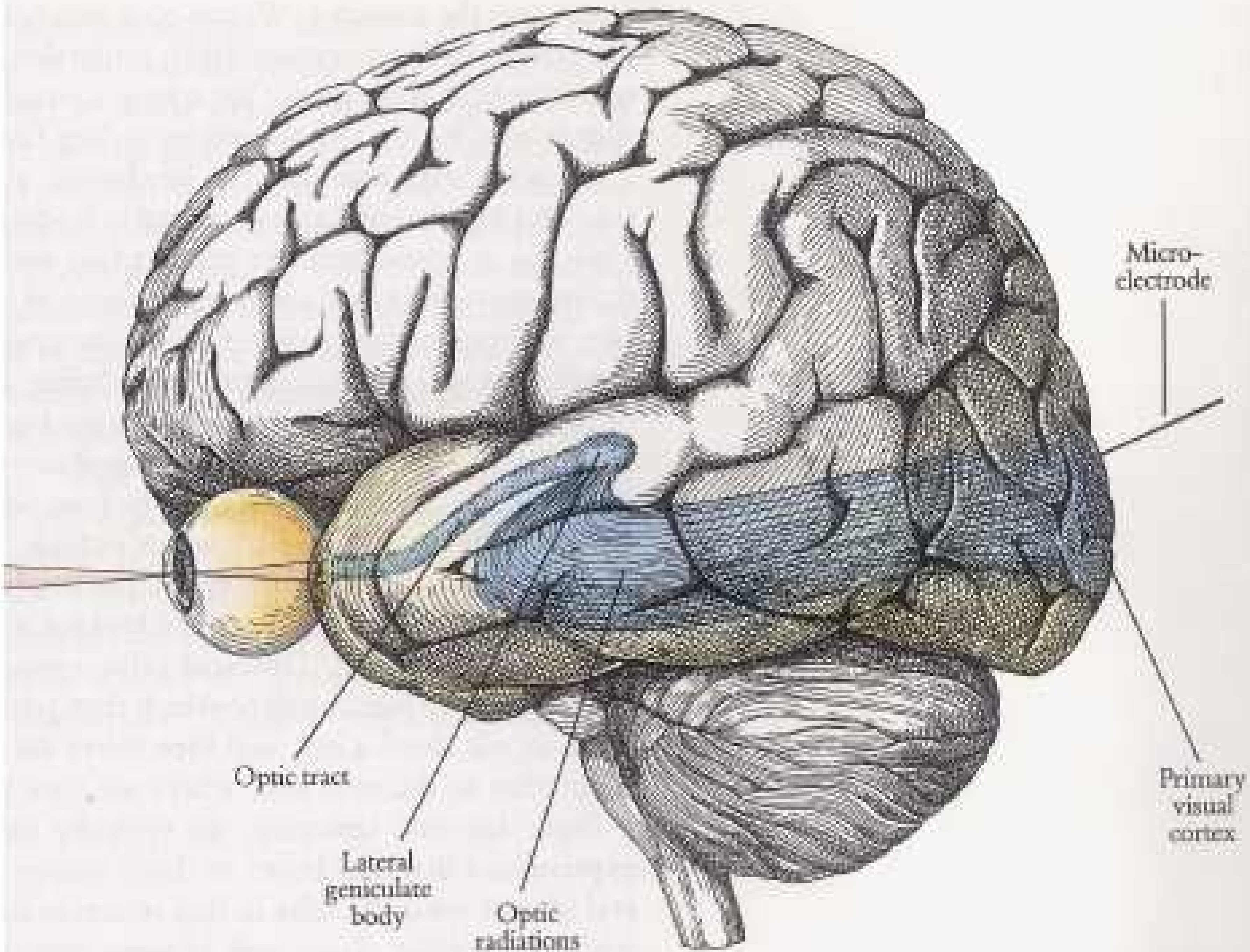
Neurogeometry
July 2 – 14, 2017
at Palazzone of the Scuola Normale Superiore, Cortona (Italy)
Organizers
Dmitri Alekseevsky
Russian Academy of Sciences
Giovanna Citti
UNIBO
Jean Petitot
CAMS-EHESS
Alessandro Sarti
CAMS-EHESS
Registration is open at http://nobelio.math.unifi.it/smi/
Deadline for applications is June 10, 2017.
Courses
First week
Enrico Le Donne
University of Jyväskylä
Riemannian and subRiemannian geometry on Lie groups
Jêrome Ribot
Collège de France
Physiological properties of neurons in the primary visual cortex
Second week
Davide Barbieri
Universidad Autonoma de Madrid
Harmonic Analysis for Modeling Visual Cortical Functions
Remco Duits
TU Eindhoven
Orientation Score Theory and its Solutions to Cortical PDE, ODE and Wavelet Models
http://germanio.math.unifi.it

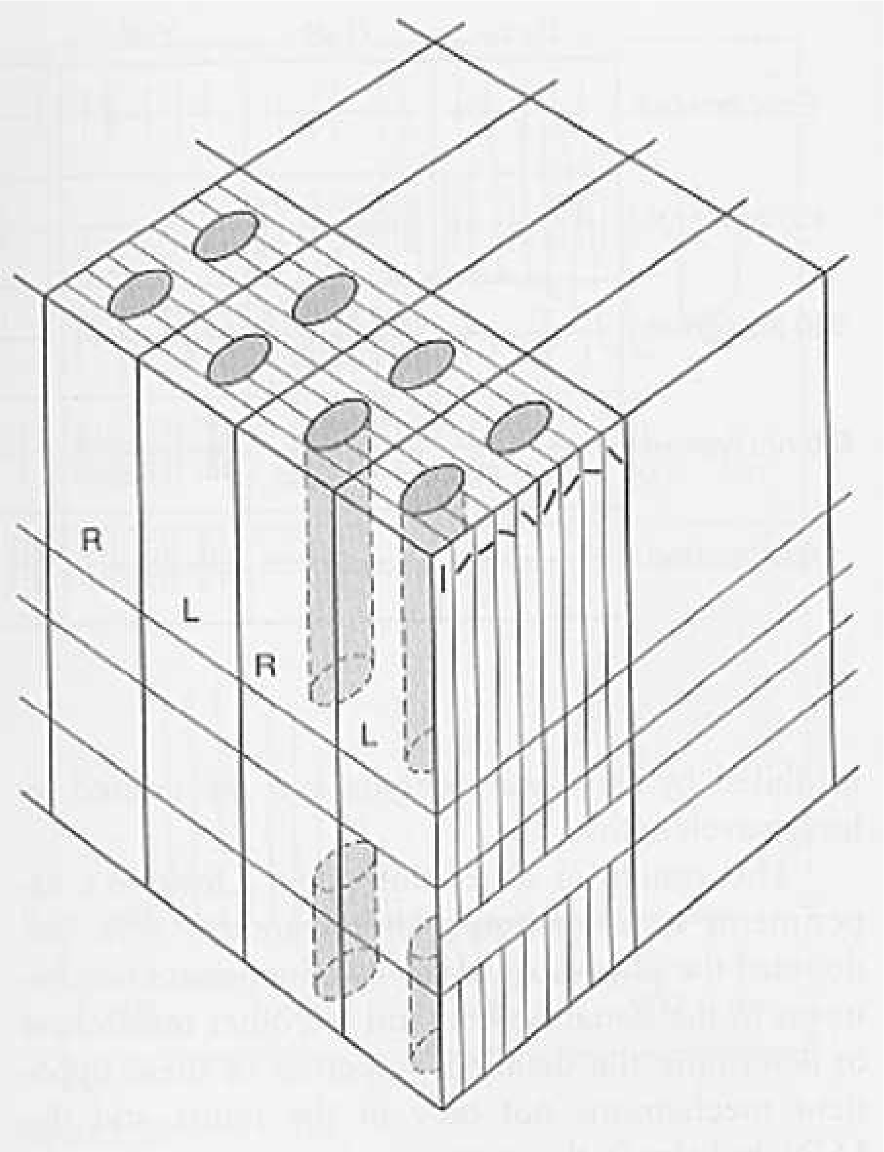
Visual cortex as a geometric engine

- Jan Koenderink: jet spaces and receptive profiles as directional derivatives (1987)
- William Hoffman : Visual cortex is a contact bundle (1989)
- S. Zucker : Frenet Frames
- Jean Petitot, Yannick Tondut : Geodesics in contact bundle (1999)
- G.Citti, A.Sarti : Principal bundle on the roto-translation group equipped with sub-Riemannian metric (2003)
- other groups (Affine, Galilean, Engel) (2006-present)
- Remco Duits image analysis on Lie groups

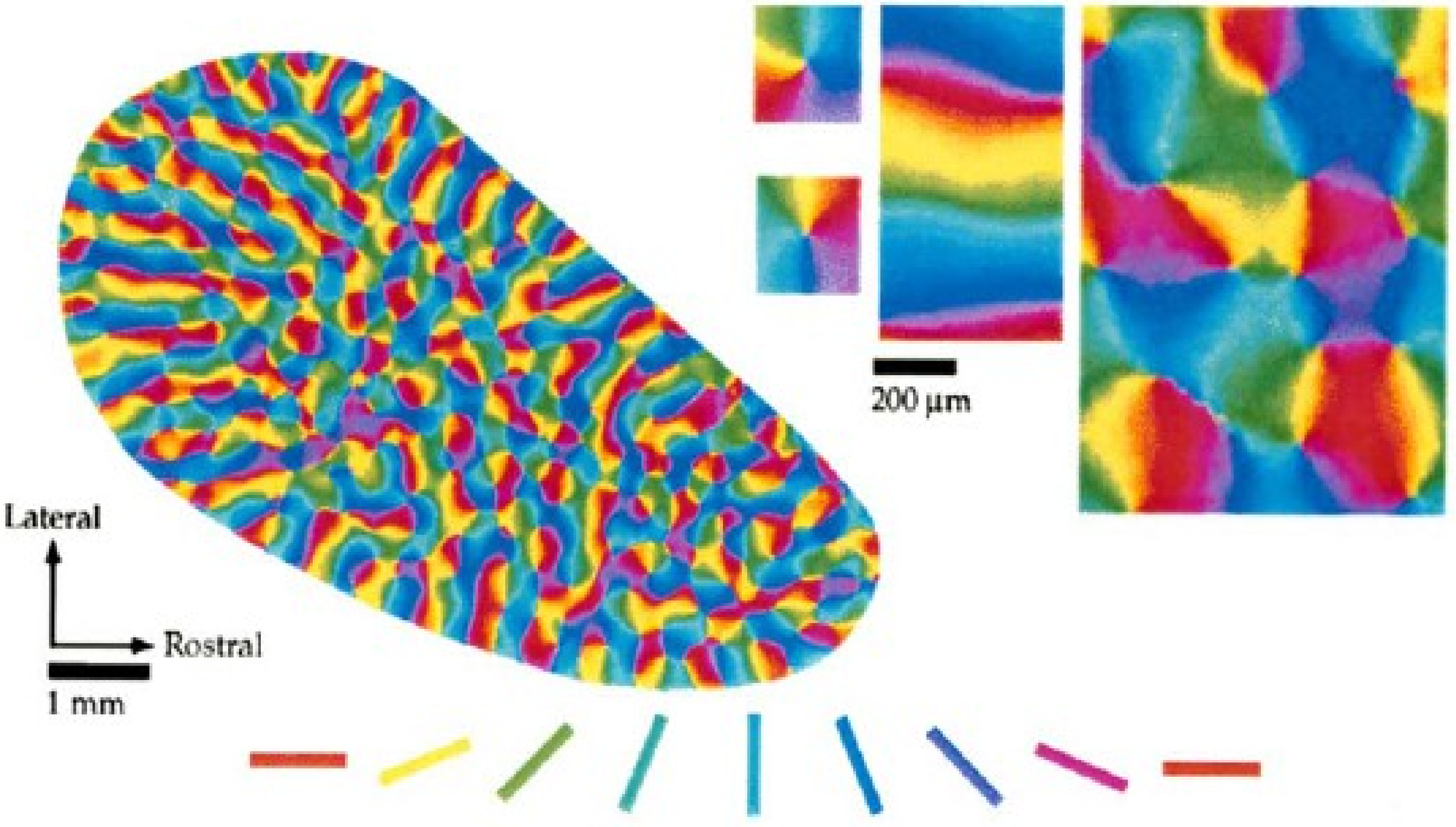


Micro-
electrode
Optic tract
Lateral
geniculate
body
Optic
radiations
Primary
visual
cortex

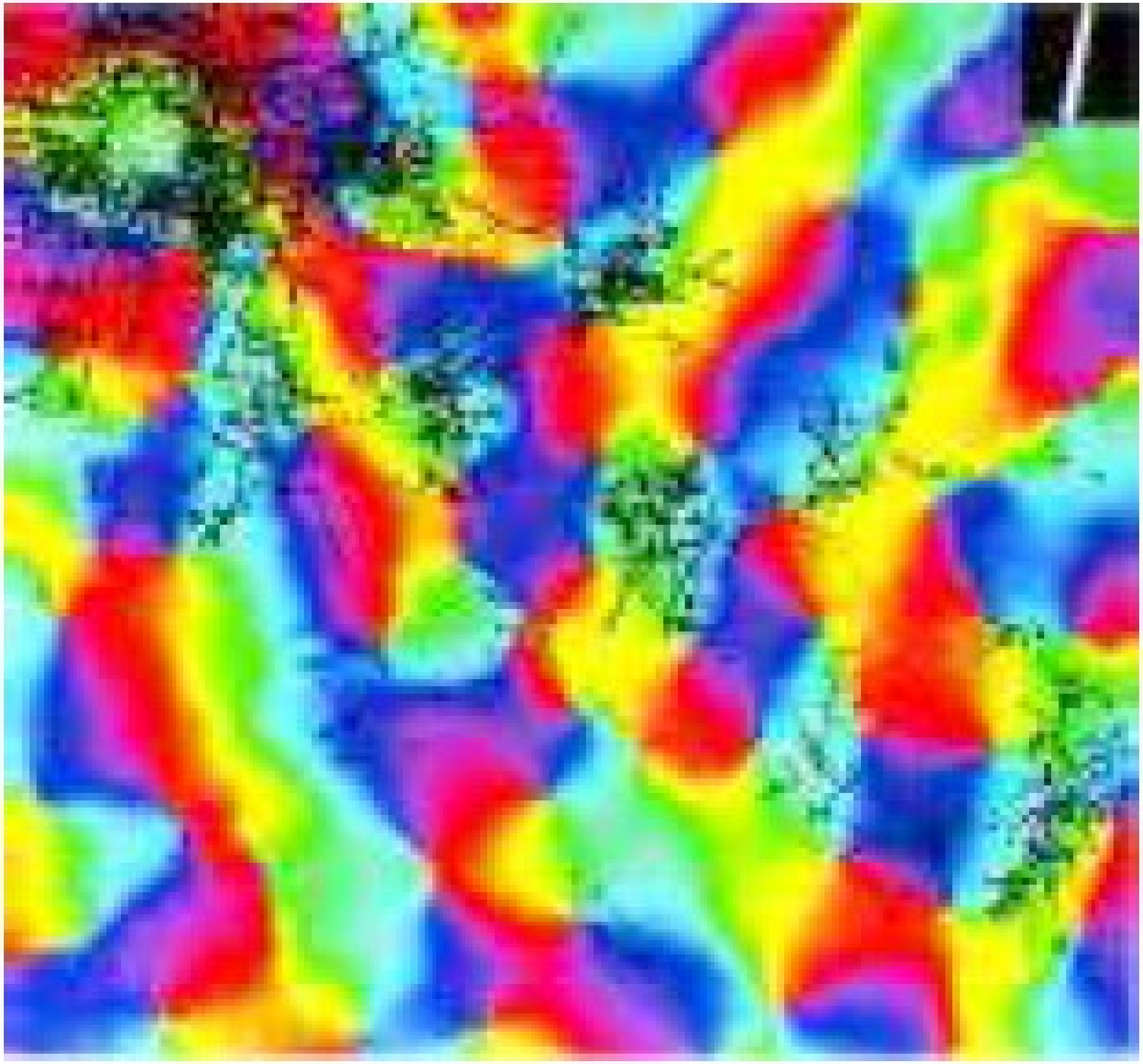
The hypercolumnar module



The pinwheel structure

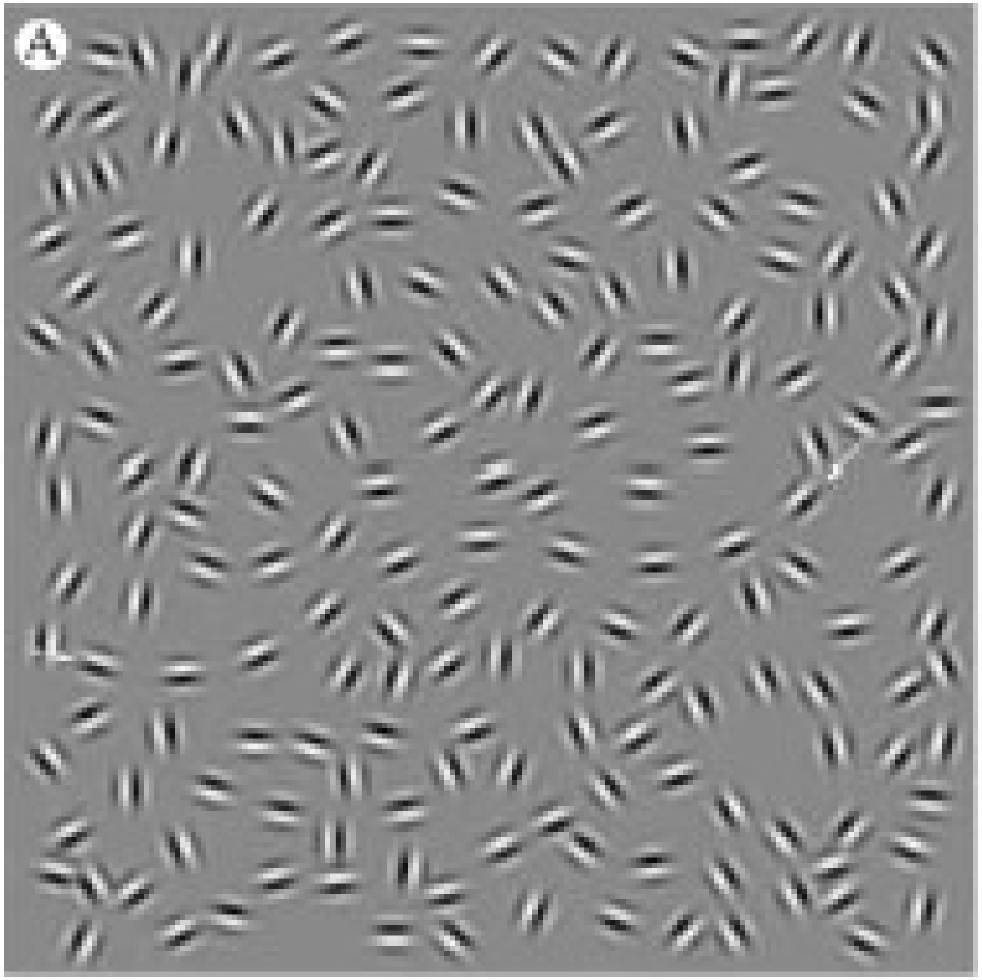


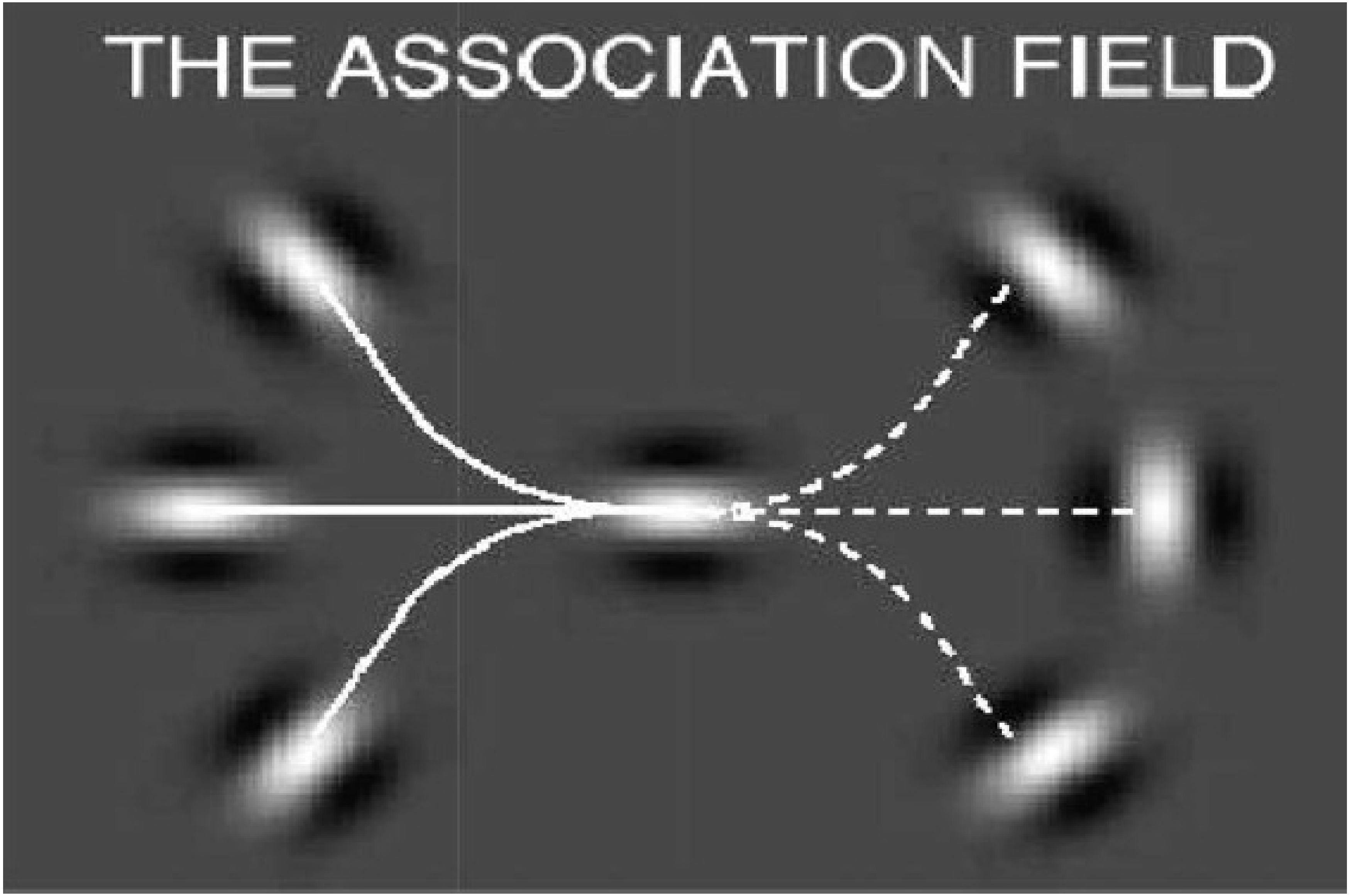
Cortico-cortical connectivity



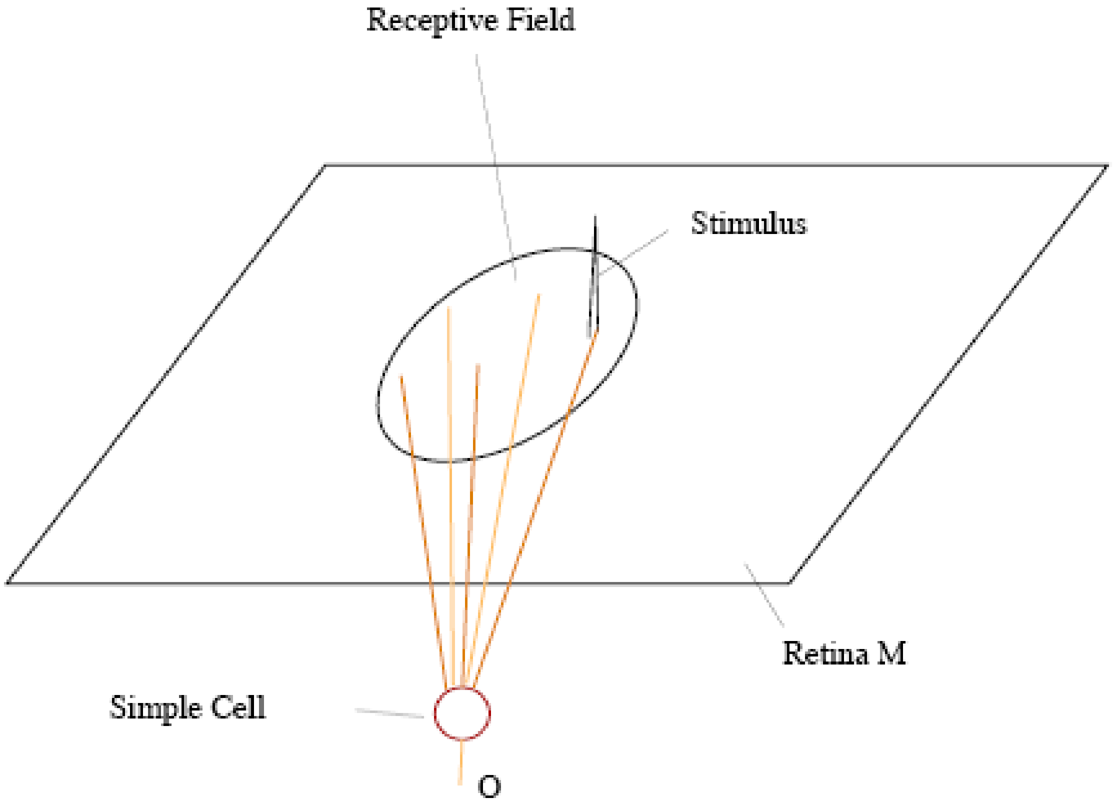
W.H. Bosking 1997

Association Field Experiment
Field, Heyes, Hess, 1993




THE ASSOCIATION FIELD

Receptive Fields and Receptive Profiles



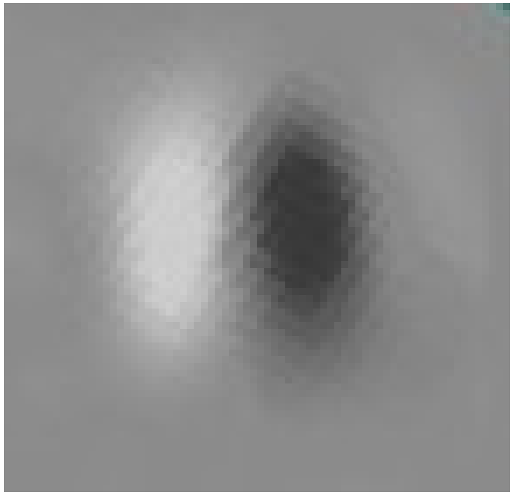
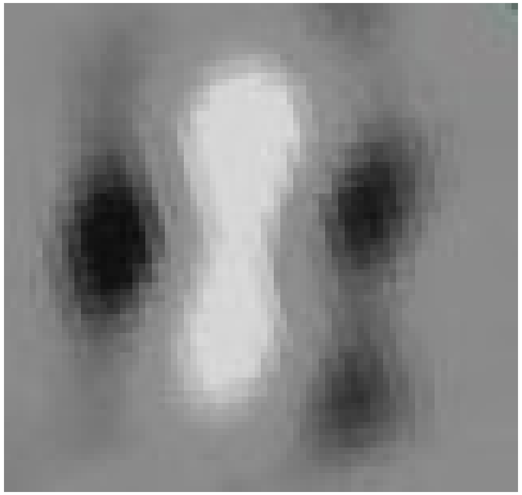
Retina $(\xi, \eta) \in M \subset \mathbb{R}^2$

Stimulus $I : M \rightarrow \mathbb{R}^+$

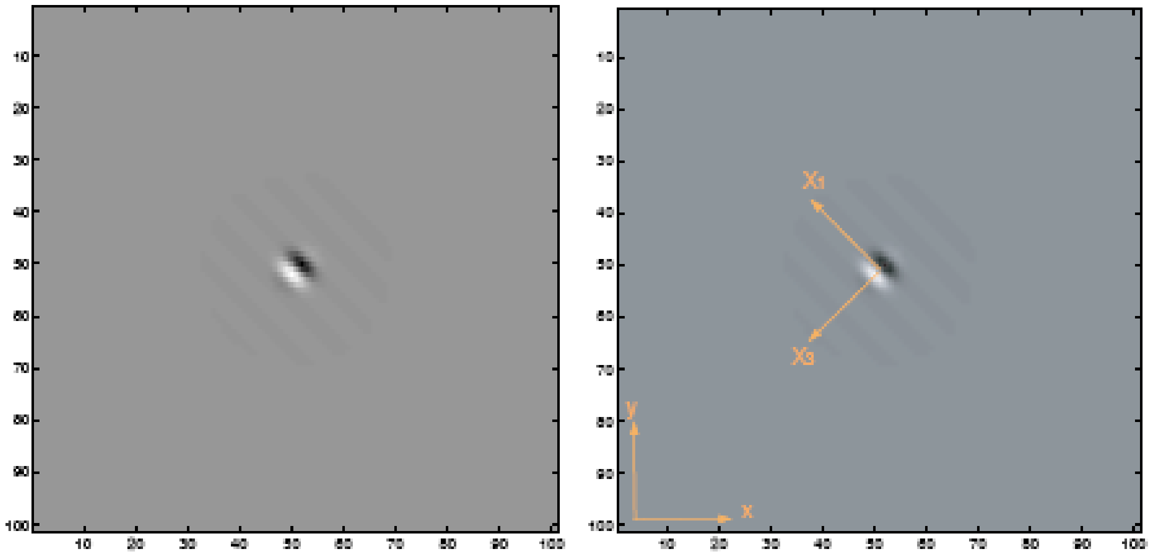
Output $O = O(I)$

Receptive profile $G(\xi, \eta) = O(\delta(\xi - \chi, \eta - \zeta))$

Simple cells receptive profiles

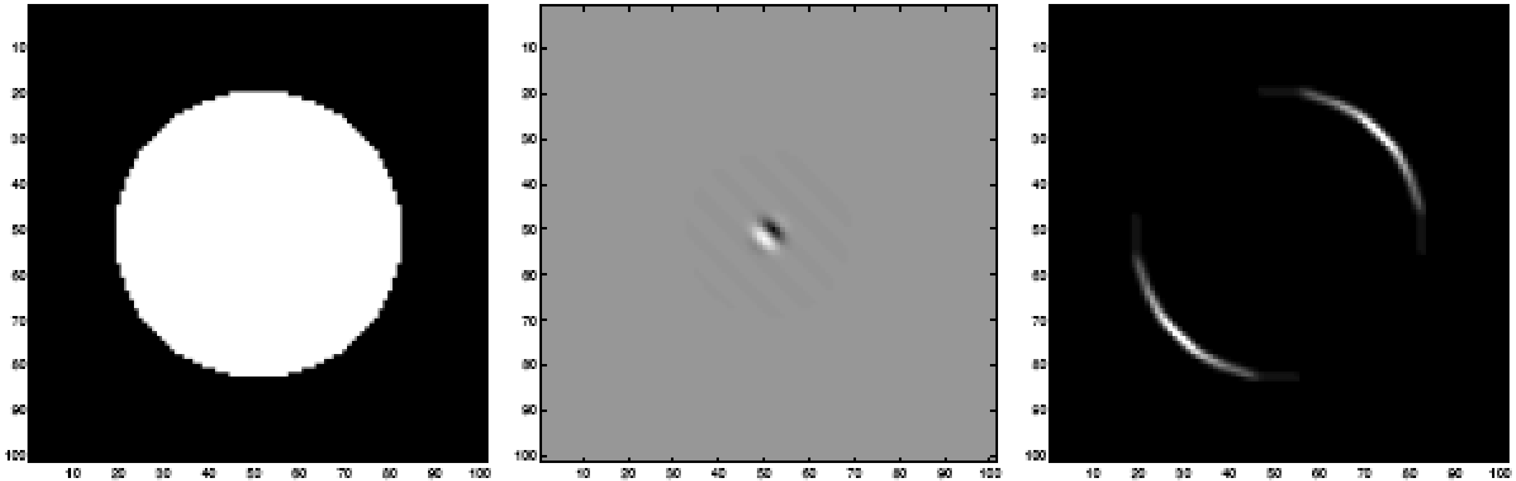
Simple Cells Receptive Profiles



$$G_{(\bar{x},\bar{y},\bar{\theta})}(\xi,\eta) = X_3(\bar{\theta})\frac{e^{-\frac{(\bar{x}-\xi)^2+(\bar{y}-\eta)^2}{s}}}{s}$$

$$X_3(\bar{\theta}) = -\sin\bar{\theta}\partial_\xi + \cos\bar{\theta}\partial_\eta$$

Cell response to an image



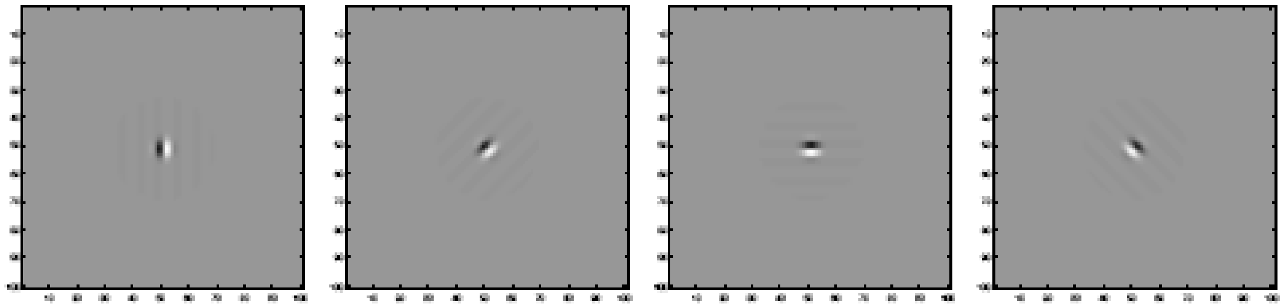
$I : M \subset \mathbb{R}^2 \rightarrow \mathbb{R}^+$ $\qquad G_{(\bar{x},\bar{y},\bar{\theta})}$ $\qquad O(x, y, \bar{\theta})$

$$O(x, y, \bar{\theta}) = \int G_{(x,y,\bar{\theta})}(\xi, \eta) I(\xi, \eta) d\xi d\eta =$$

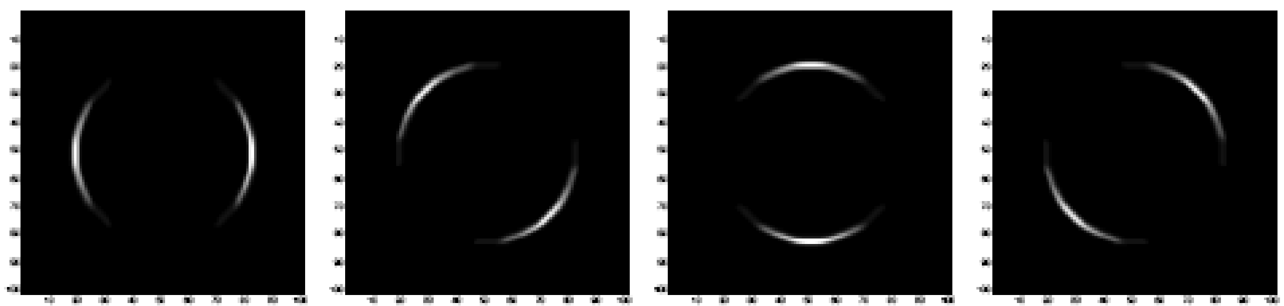
$$X_3(\bar{\theta}) \frac{e^{-\frac{x^2+y^2}{s}}}{s} \star I = X_3(\bar{\theta}) I_s$$

Hypercolumn as fiber of derivations

$$F(\bar{x}, \bar{y}) = \{G(\bar{x}, \bar{y}, \theta), \theta \in [0, \pi]\}$$

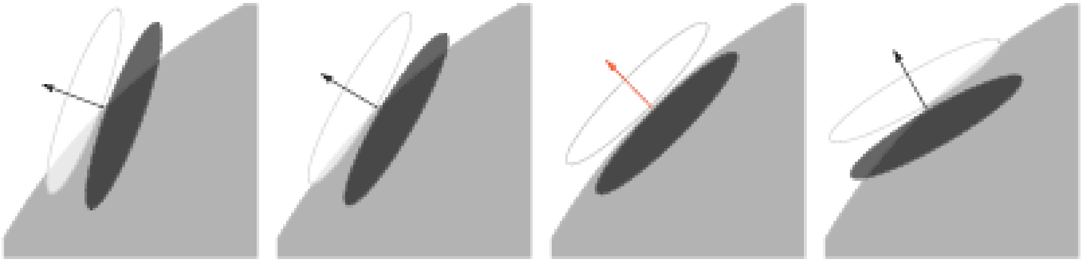


$$O(x, y, \theta) = \int G_{(x,y,\theta)}(\xi, \eta) I(\xi, \eta) d\xi d\eta = X_3(\theta) I_s$$

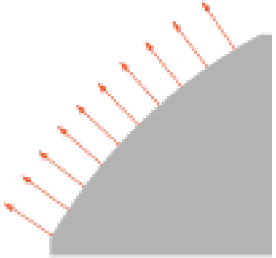


$$X_3(\theta) I_s = (-\sin\theta \partial_x + \cos\theta \partial_y) I_s$$

Maximum Response Selection

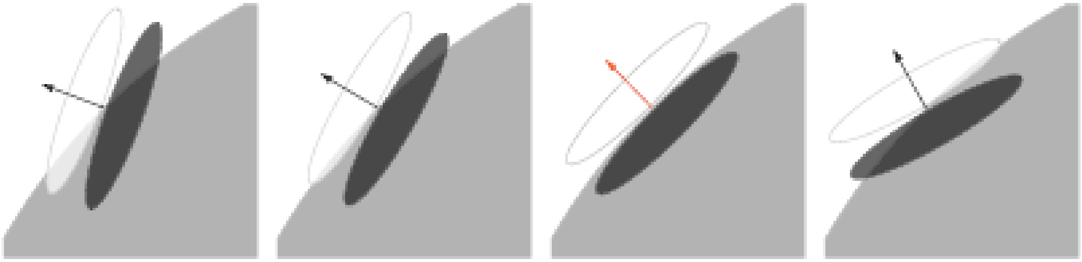


$$O(\bar{x}, \bar{y}, \tilde{\theta}) = \max_{\theta \in [0,\pi]} O(\bar{x}, \bar{y}, \theta)$$

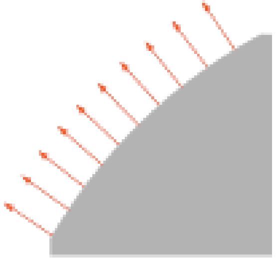


$$O(x, y, \tilde{\theta}(x, y)) = \max_{\theta \in [0,\pi]} O(x, y, \theta)$$

Maximum Response Selection

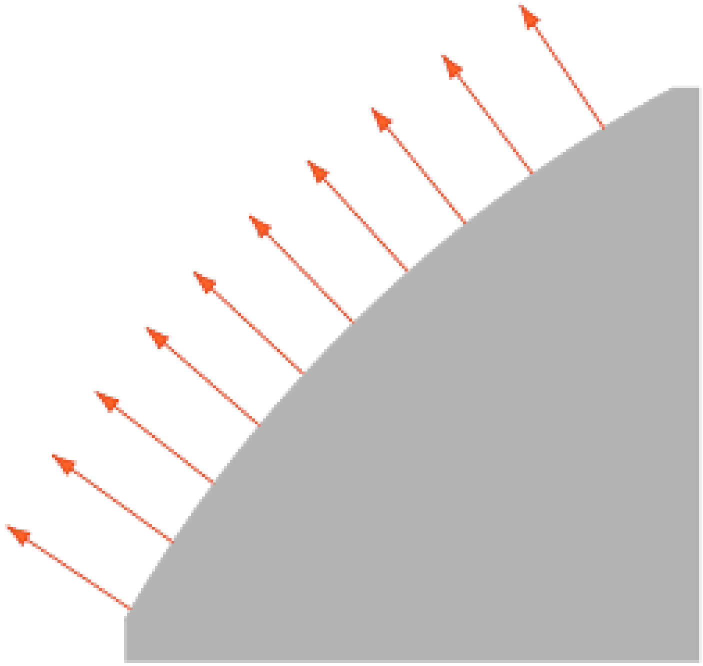


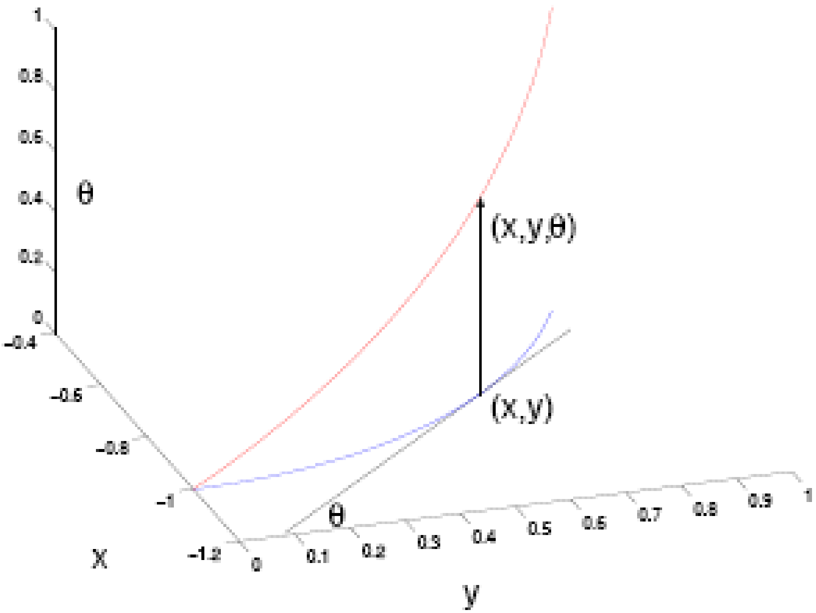
$$O(\bar{x}, \bar{y}, \tilde{\theta}) = \max_{\theta \in [0, \pi]} O(\bar{x}, \bar{y}, \theta)$$



$$O(x, y, \tilde{\theta}(x, y)) = \max_{\theta \in [0, \pi]} O(x, y, \theta)$$

Geometric Interpretation: lifting





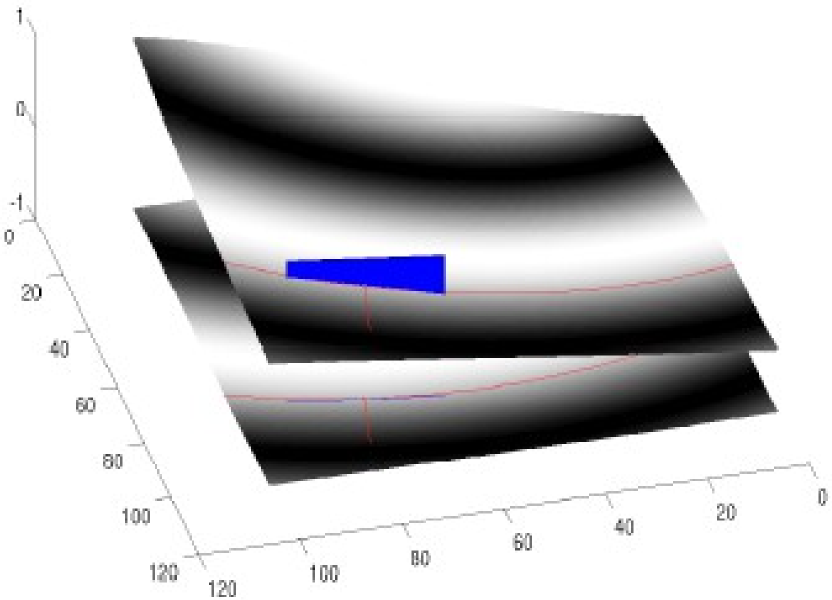
$$\partial_\theta O(x,y,\theta) = \partial_\theta (X_3(\theta) I_s) = -X_1(\tilde{\theta}(x,y)) I_s(x,y) = 0$$

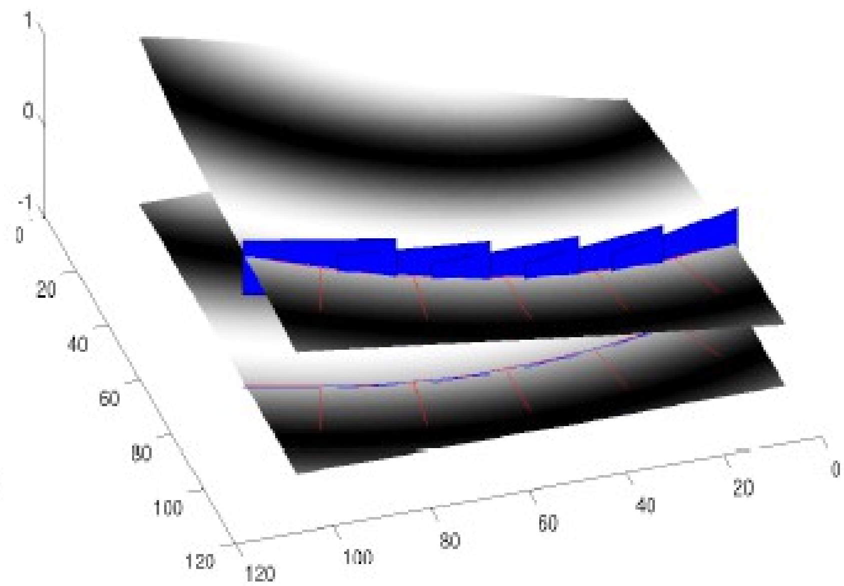
$$(\cos\tilde{\theta}\partial_x + \sin\tilde{\theta}\partial_y) I_s(x,y) = 0$$

$$(-\sin\tilde{\theta}dx + \cos\tilde{\theta}dy)(\cos\tilde{\theta}\partial_x + \sin\tilde{\theta}\partial_y) = 0$$

$$< \omega_3, X_1 >= 0$$

Lifting in the rototranslation space

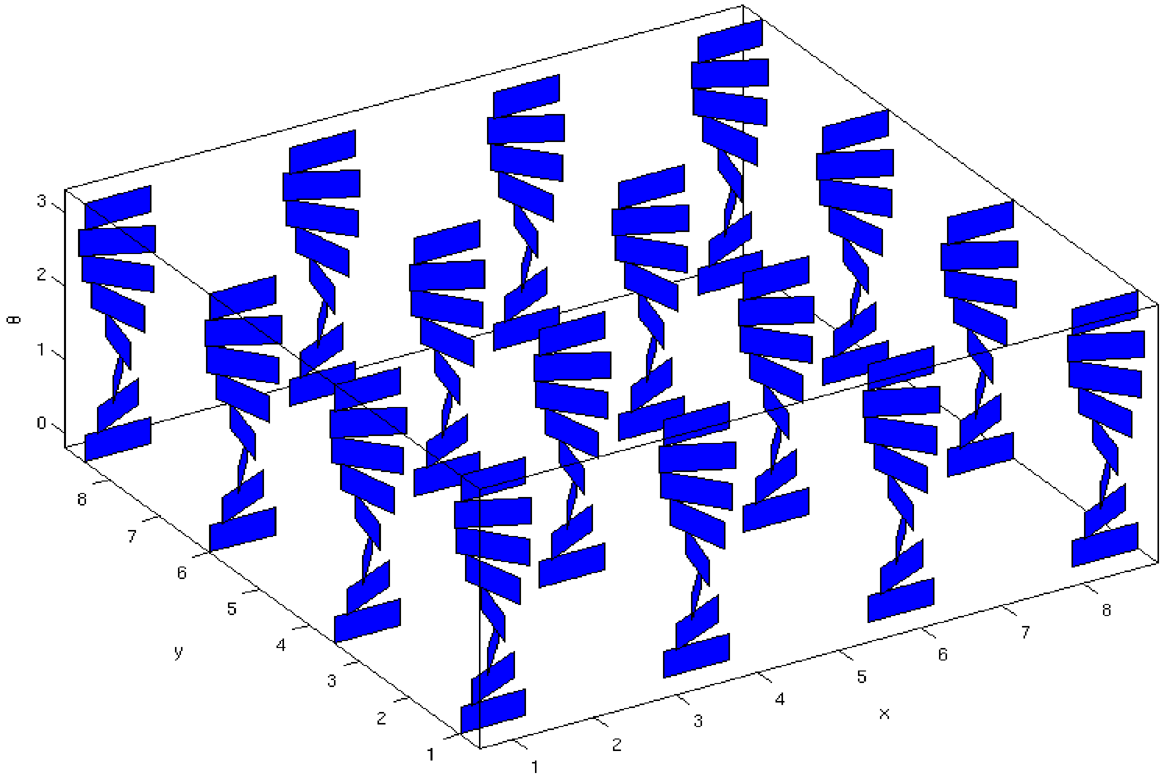




$$\vec{X}_1 = (\cos(\theta), \sin(\theta), 0)$$

$$\vec{X}_2 = (0, 0, 1)$$

Tangent planes



$$\vec{X}_1 = (\cos(\theta), \sin(\theta), 0)$$

$$\vec{X}_2 = (0, 0, 1)$$

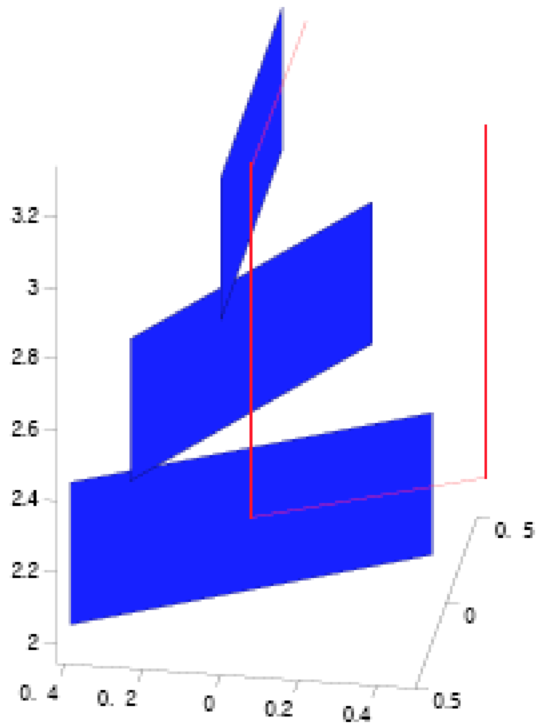
$$\alpha = -\sin(\theta)dx + \cos(\theta)dy$$

G.Citti, A.Sarti 2003,2006

The rototranslation algebra

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad X_2 = \partial_\theta$$

$$X_3 = [X_2, X_1] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$



Left invariant vector fields $X_{i,g}(f(g_1 \circ g)) = (X_{i,g}f)(g_1 \circ g)$

The sub-Riemannian metric

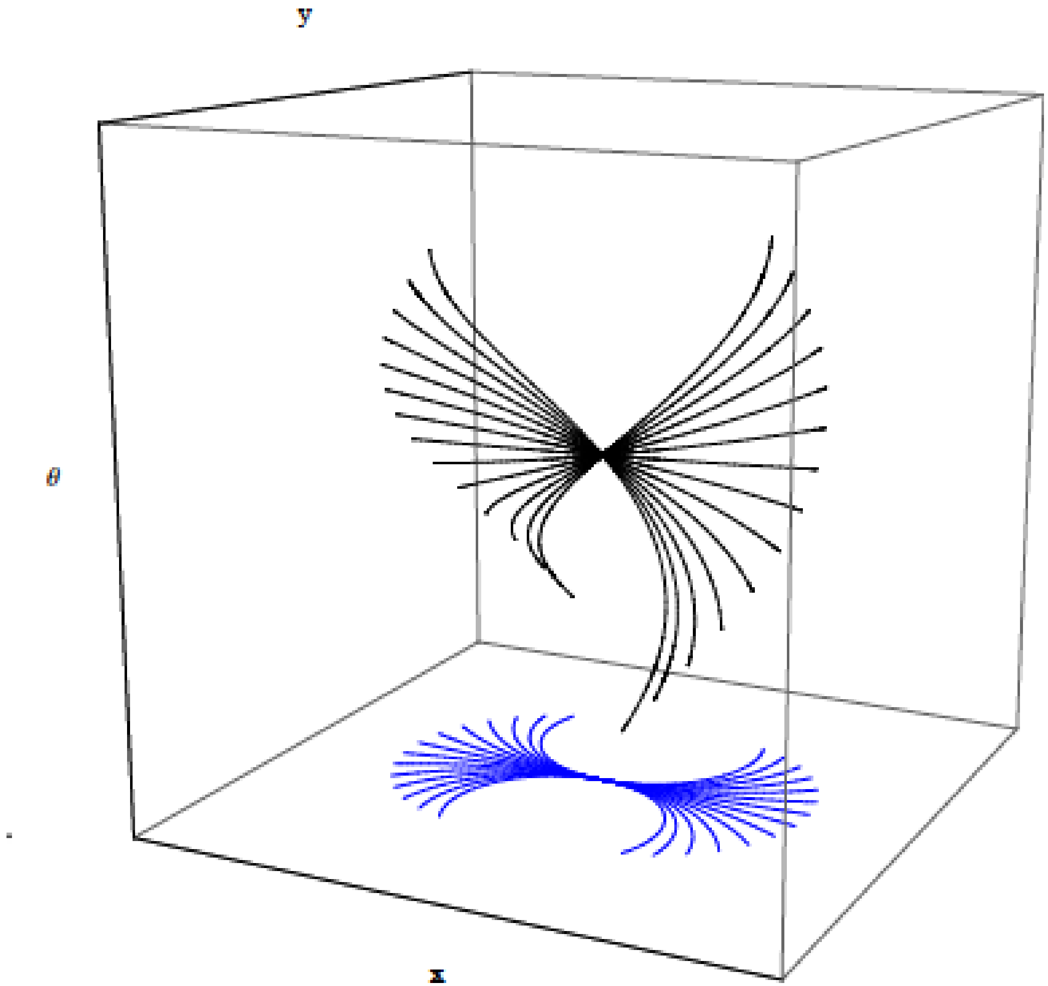
On the contact planes we consider an Euclidean metric

$$c^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then the space metric is, by change of coordinates

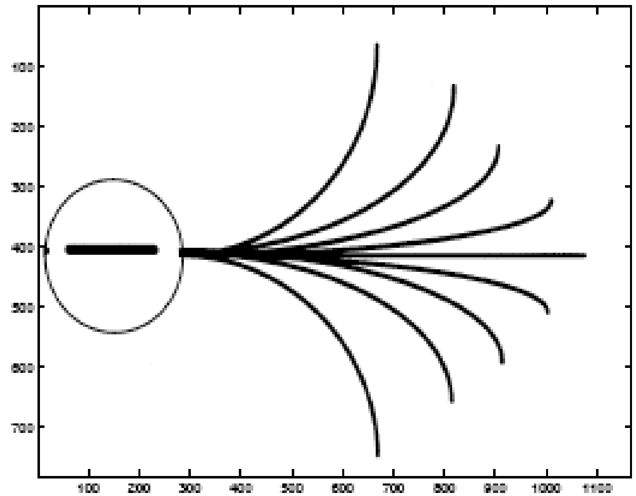
$$g^{ij} = \begin{bmatrix} cos\theta & 0 \\ sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} cos^2\theta & cos\theta sin\theta & 0 \\ cos\theta sin\theta & sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

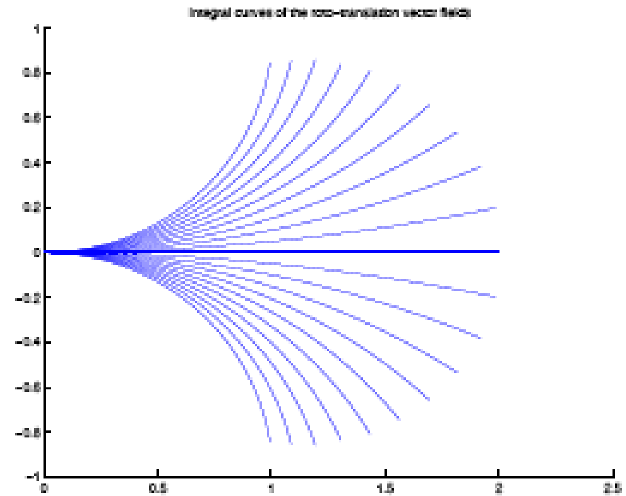
Integral curves



$$\gamma'(t) = \vec{X}_1(\gamma) + k\vec{X}_2(\gamma), \quad \gamma(0) = (x, y, \theta).$$

Association Fields and Integral Curves of Tangent Vectors



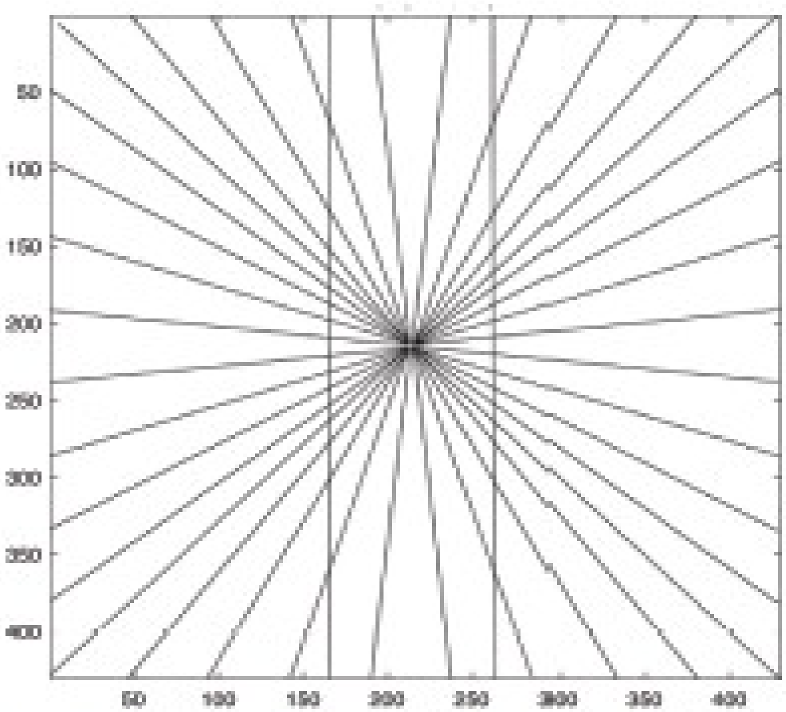


$$\gamma'(t) = (\gamma_1 \vec{X}_1 + \gamma_2 \vec{X}_2) \quad \gamma(0) = (x, y, \theta)$$

$$\vec{X}_1 = (\cos(\theta), \sin(\theta), 0)$$

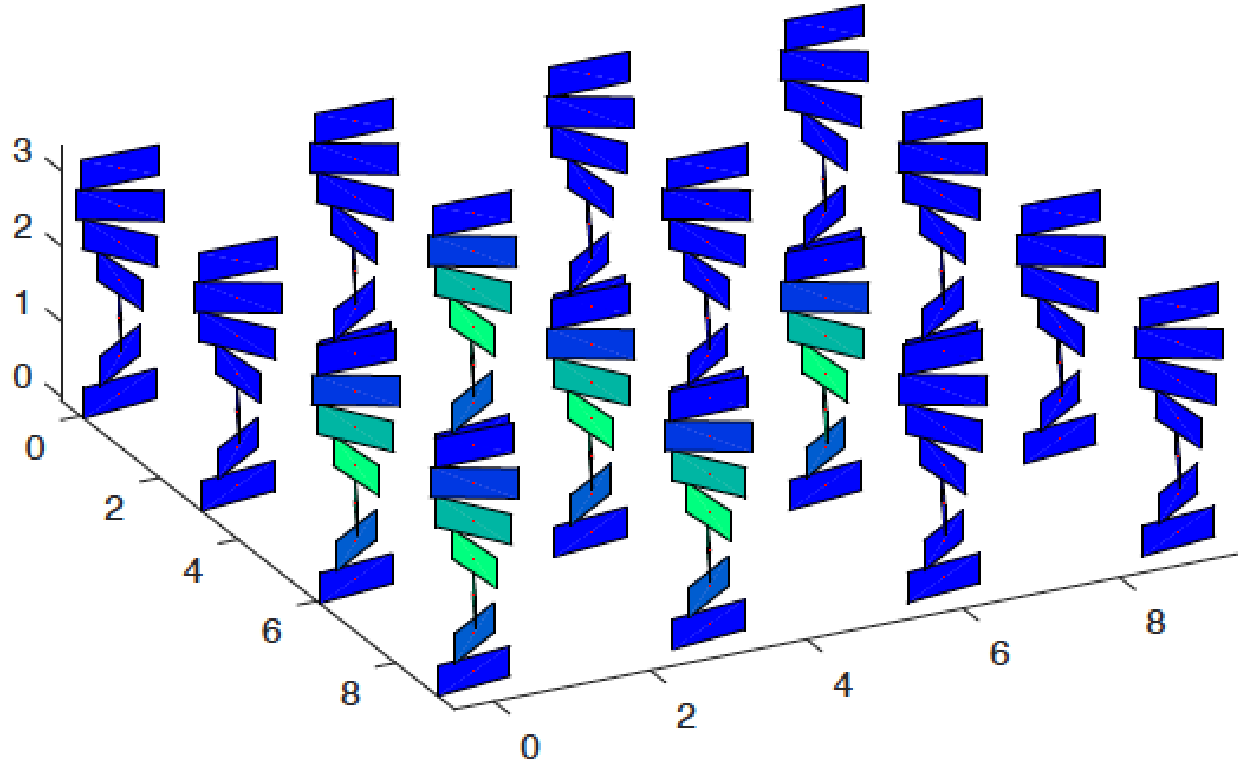
$$\vec{X}_2 = (0, 0, 1)$$

Geometric illusions



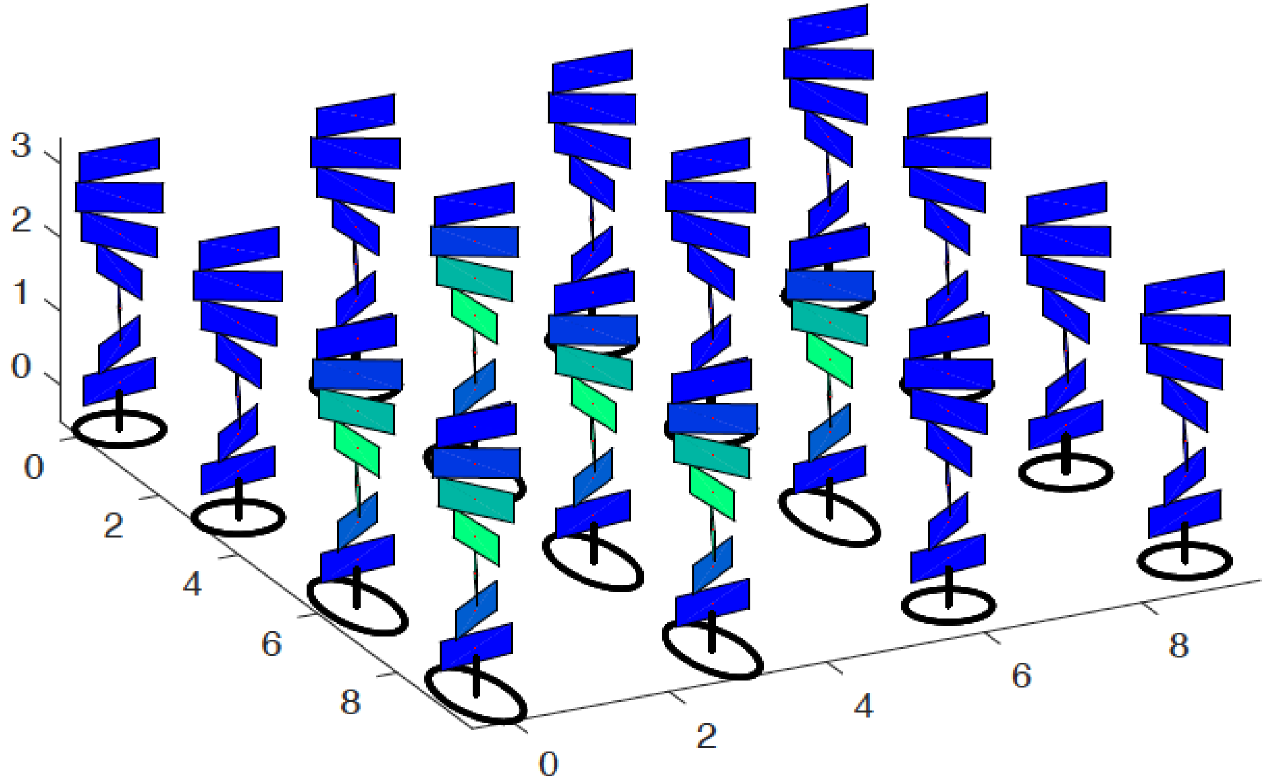
(a) Hering illusion, distal stimulus

Tangent planes with metric modulation



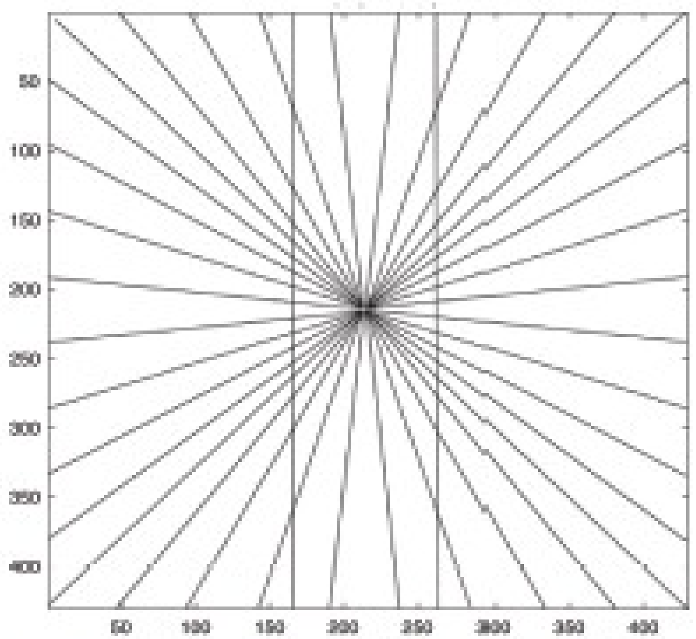
$$g^{ij}(x_1, x_2, \theta) = (1 + O(x_1, x_2, \theta)) \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta & \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Projection onto the image plane

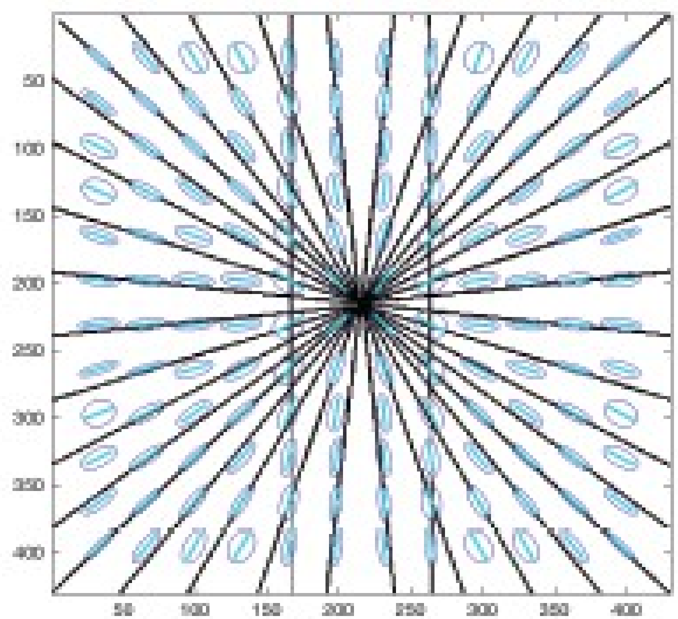


$$g_I^{ij}(x_1, x_2) = \int_\theta (1 + O(x_1, x_2, \theta)) \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} d\theta$$

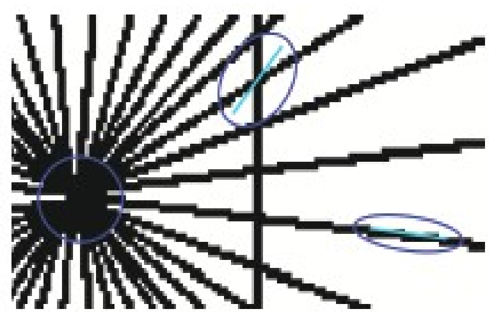
Hering Illusion



(a) Hering illusion, distal stimulus



(b) Tensor representation



B.Franceschiello, G.Citti, A.Sarti 2017

Infinitesimal strain/displacement

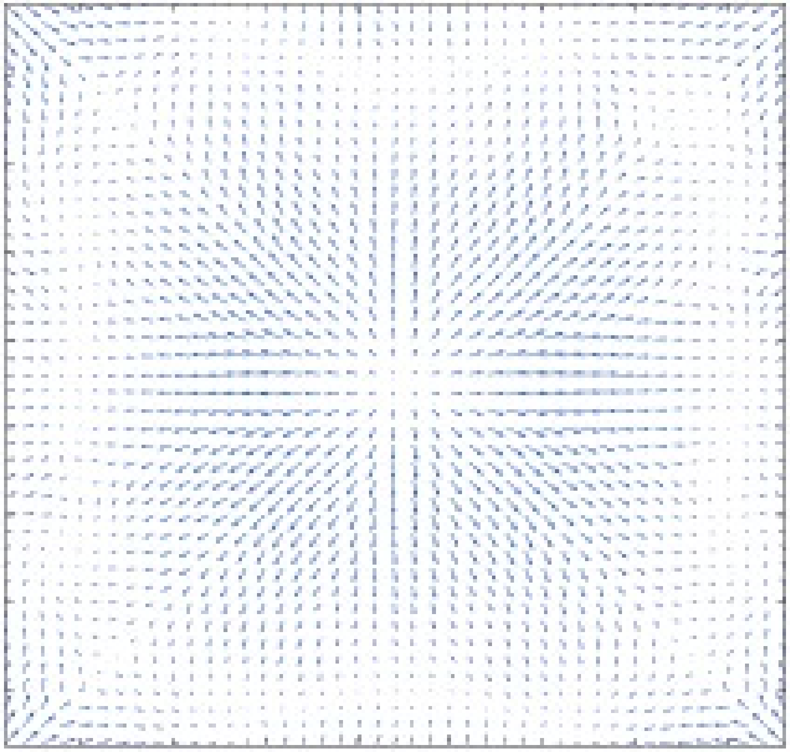
$$(g - Id)(x_1, x_2) = \nabla u(x_1, x_2) + (\nabla u(x_1, x_2))^T$$

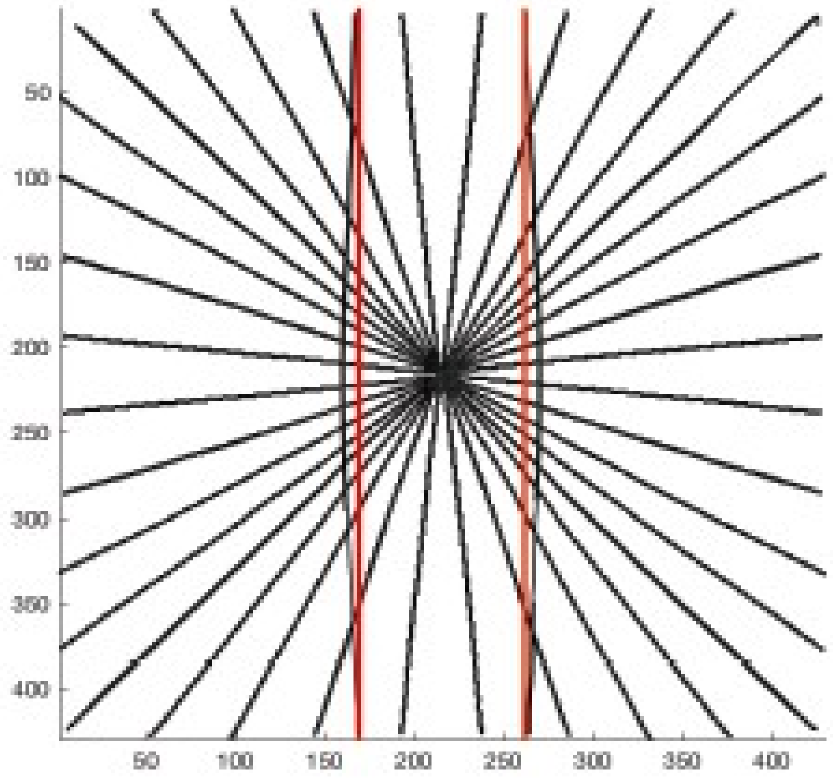
- Differentiating and substituting:

$$\begin{cases} \Delta u_1 = -\frac{\partial}{\partial x_1} g_{22} + \frac{\partial}{\partial x_1} g_{11} + 2 \frac{\partial}{\partial x_2} g_{12} \\ \Delta u_2 = -\frac{\partial}{\partial x_2} g_{22} + \frac{\partial}{\partial x_2} g_{11} + 2 \frac{\partial}{\partial x_1} g_{12} \end{cases}$$

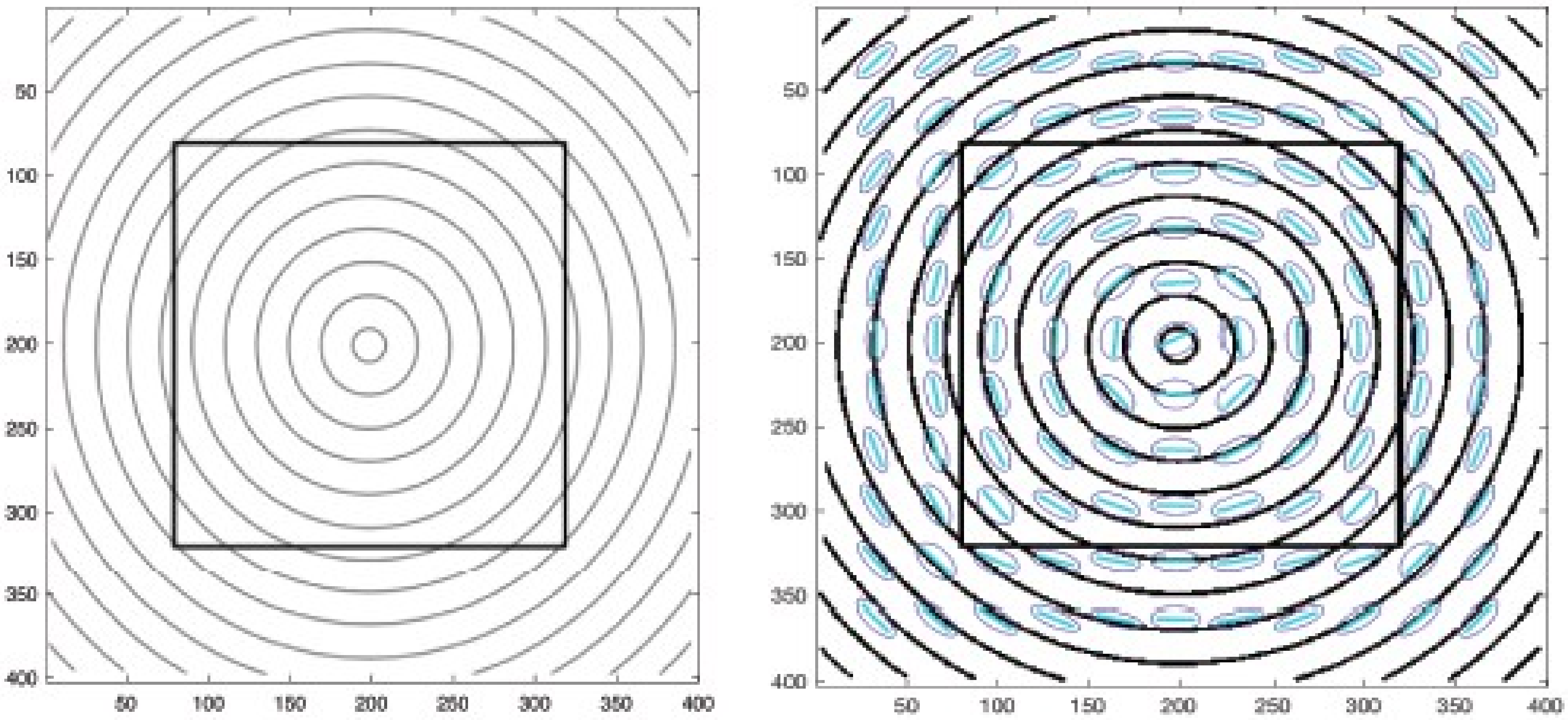
- Solving numerically the Poisson problems with Neumann boundary conditions we recover the displacement field $\{\bar{u}(x_1, x_2)\}_{(x_1, x_2) \in \mathbb{R}^2}$.

Hering Illusion

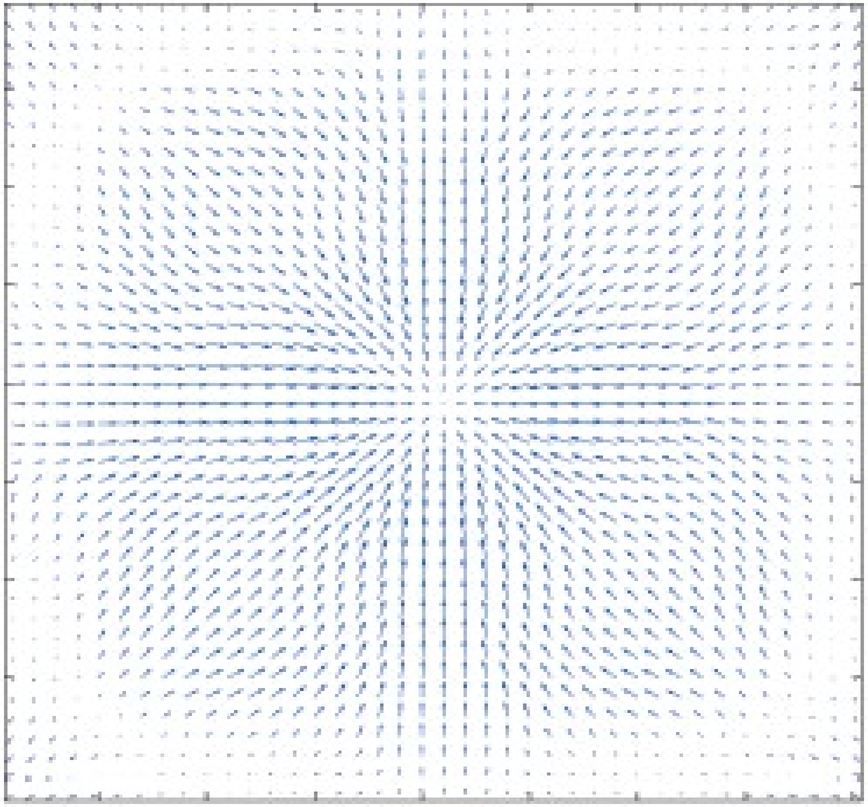


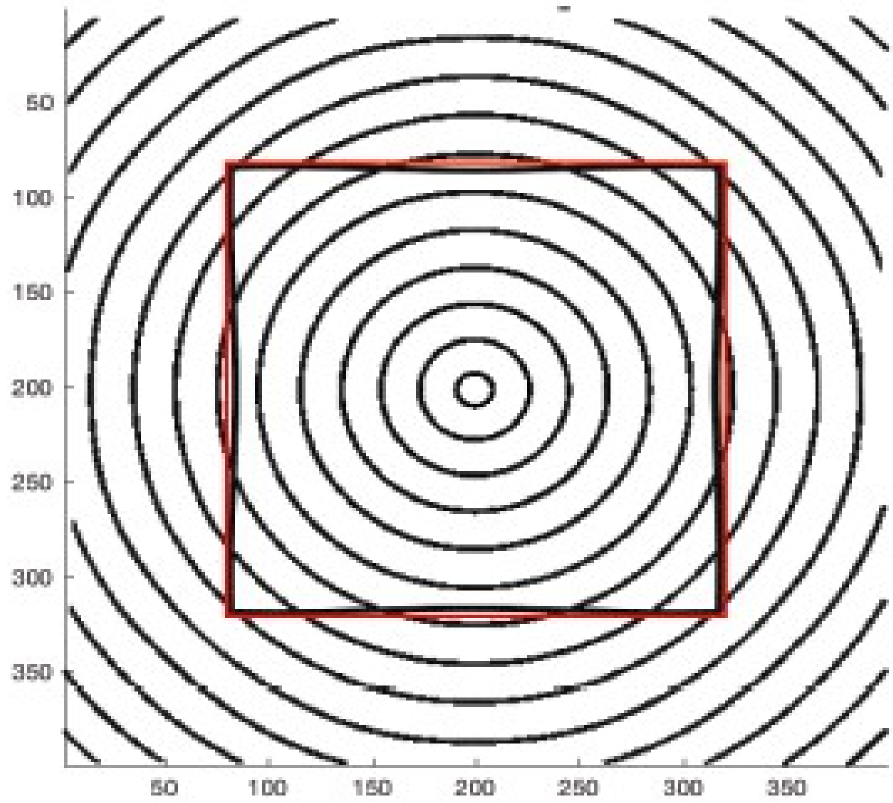


Erhenstein Illusion



Erhenstein Illusion





B.Franceschiello, G.Citti, A.Sarti. 2017

A zoo of groups

LGN Mexican hat T(2)

Simple cells SE(2) $\alpha_1 = -\sin(\theta)dx + \cos(\theta)dy$

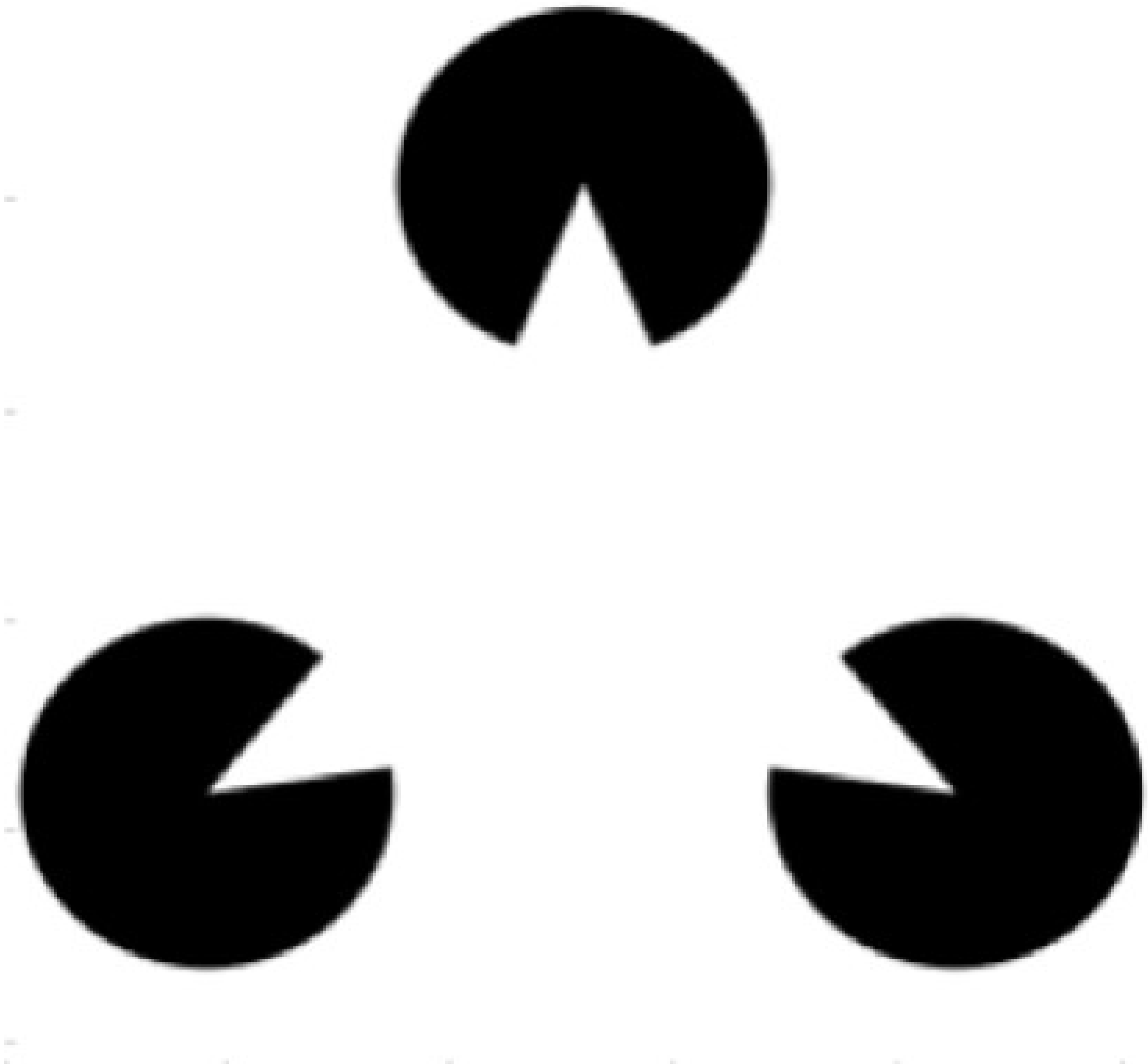
Simple cells Aff(2) $\alpha_2 = e^{-\sigma}(-\sin(\theta)dx + \cos(\theta)dy)$

Non separable complex cells Galilean $\alpha_3 = -\sin(\theta)dx + \cos(\theta)dy - vds$

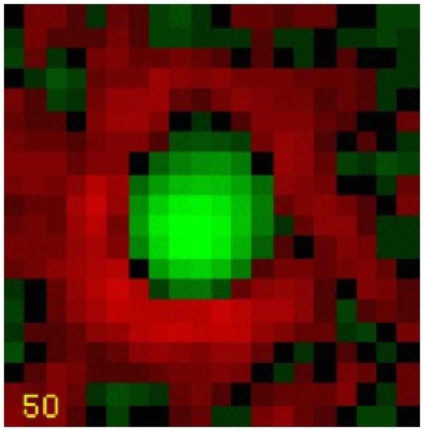
Stereo SE(3)
$$\alpha_l = -\sin(\theta_l)dx + \sin(\theta_l)d\delta + \cos(\theta_l)dy$$
$$\alpha_r = -\sin(\theta_r)dx - \sin(\theta_r)d\delta + \cos(\theta_r)dy$$

End-stopping, T-junction, L-junction complex cells

Talamo-cortical heterogeneity: SE(2) and T(2)



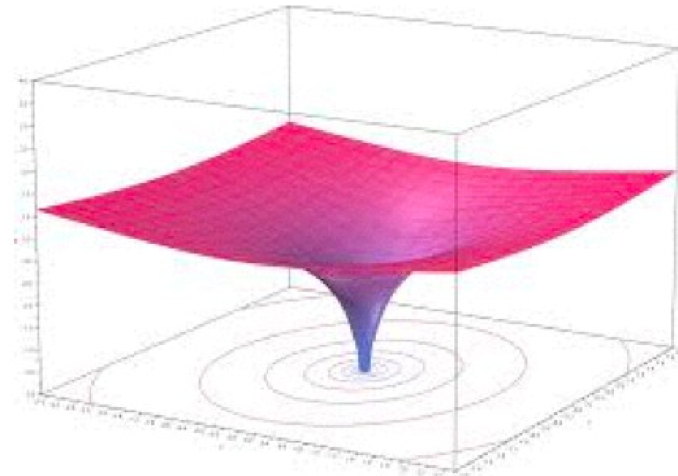
LGN



$$\Delta e^{-|x|^2}$$

$$Out = \Delta e^{-|x^2|} \star h \simeq \Delta h$$

V1->LGN



$$\Gamma : \Delta\Gamma = \delta$$

$$\phi = \Gamma \star Out$$

LGN->V1->LGN

$$\Delta\phi = \Delta h$$

$h(x,y)$

Δh

$\Delta \phi = \Delta h$

$$\int |\nabla \phi - \nabla h|^2 dxdy$$

The modal completion sub-Riemannian Lagrangian

$$\int |\nabla\phi - \nabla h|^2 dxdy + \int |\nabla\phi - \vec{A}|^2 dxdy + \int |X_1 \vec{A}|^2 dxdy$$

The Euler-Lagrange Equation

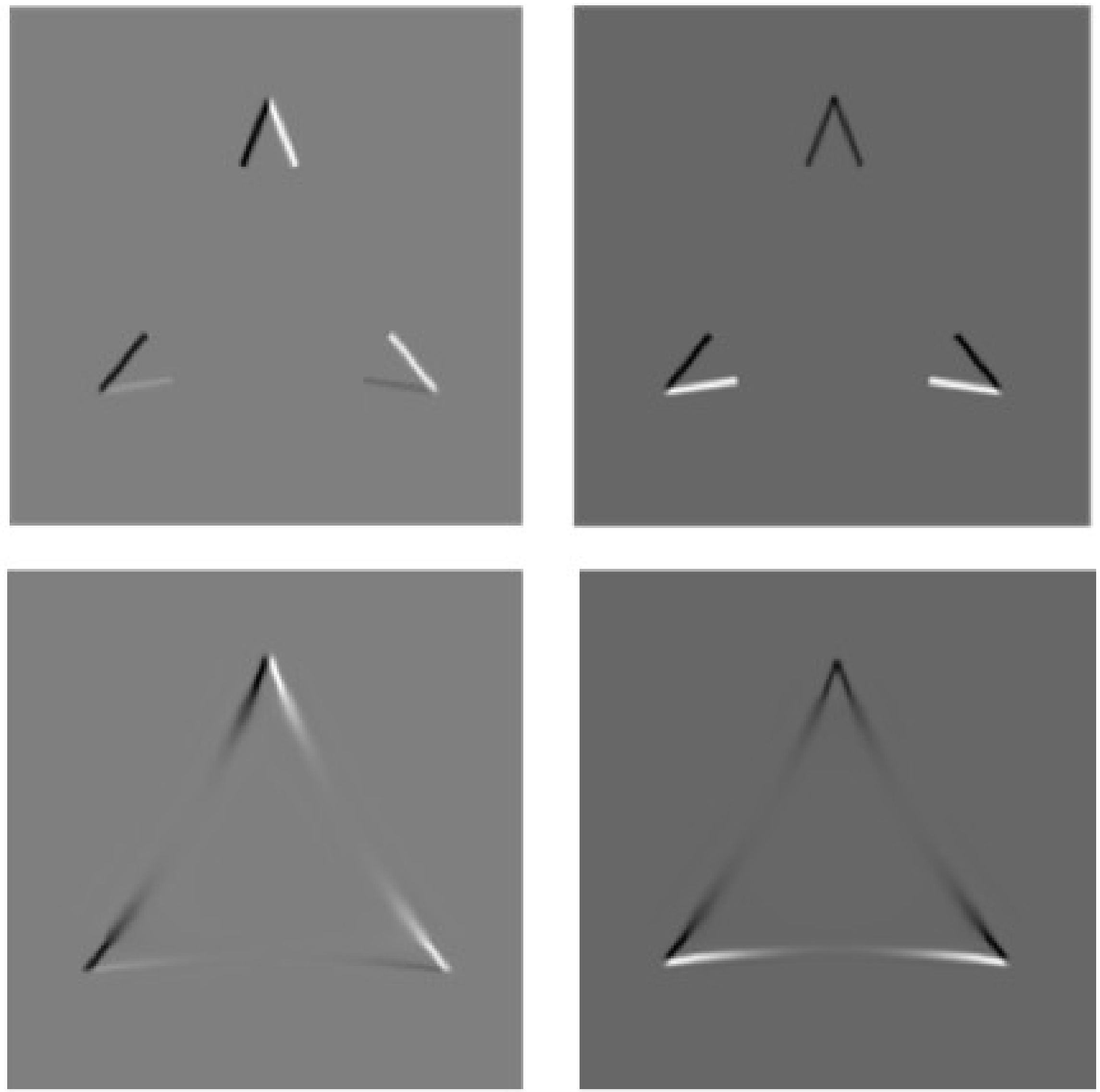
$$\Delta\phi = \frac{1}{2}(\Delta h + div(\vec{A}))$$
$$X_{11}\vec{A} = -\nabla\phi + \vec{A}$$

$$h \to h + k \qquad \phi \to \phi + k \qquad \vec{A} \to \vec{A} + \nabla k$$

G.Citti, A.S. 2014

The field term

$$X_{11}\vec{A} = -\nabla\phi$$



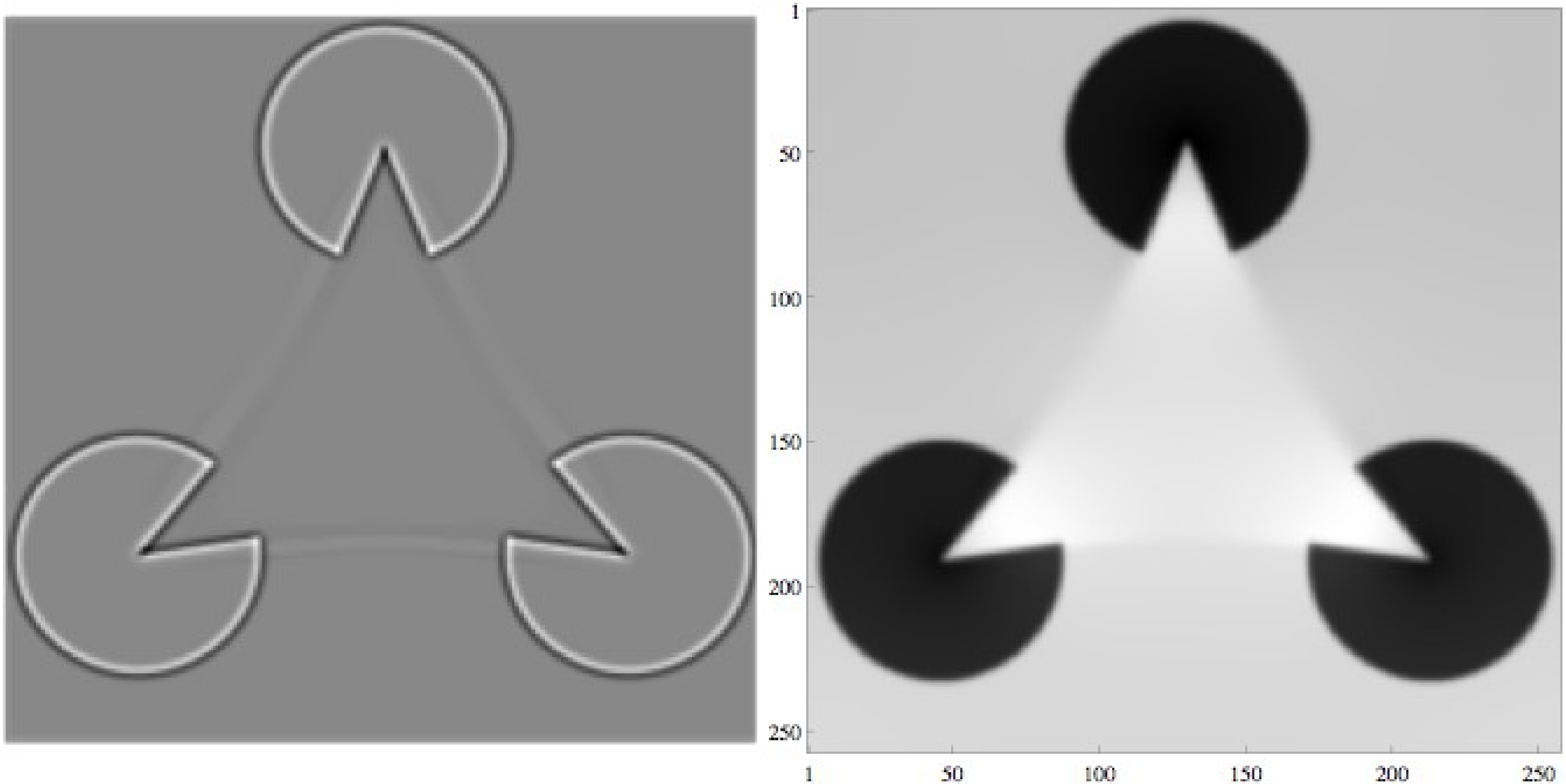
The particle term

$$\Delta\phi = \frac{1}{2}(\Delta h + div(\vec{A}))$$

$$\phi = \Gamma_{LGN} \star (feedf_{LGN} + horiz_{V1})$$

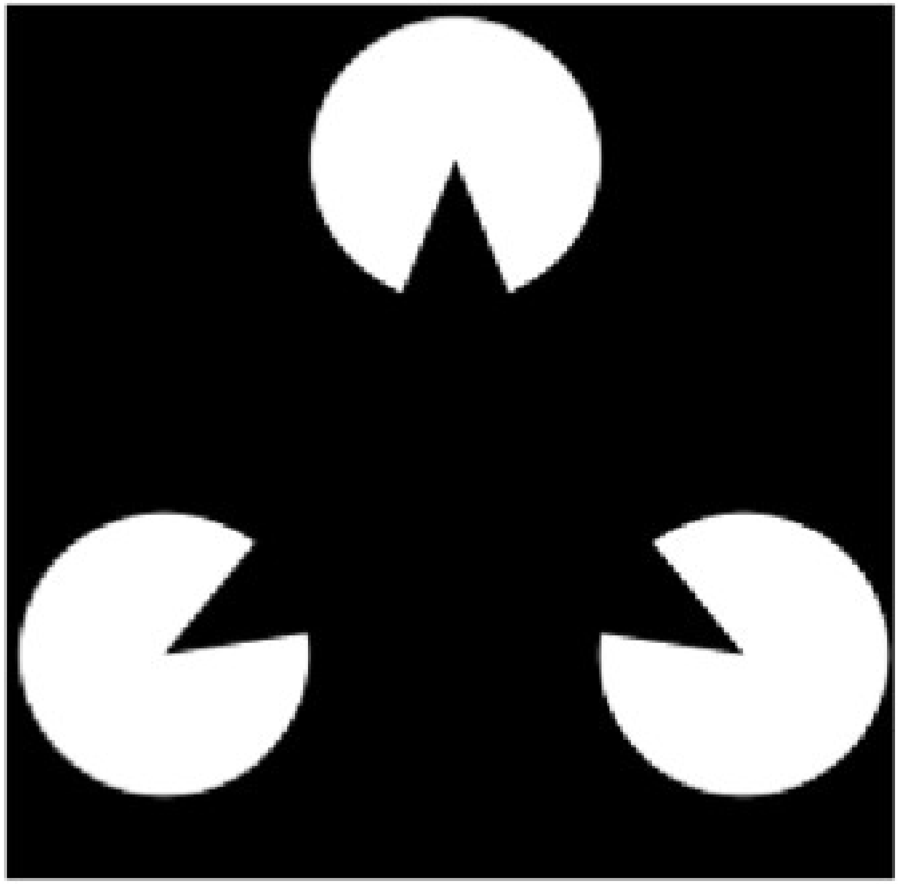
The particle term

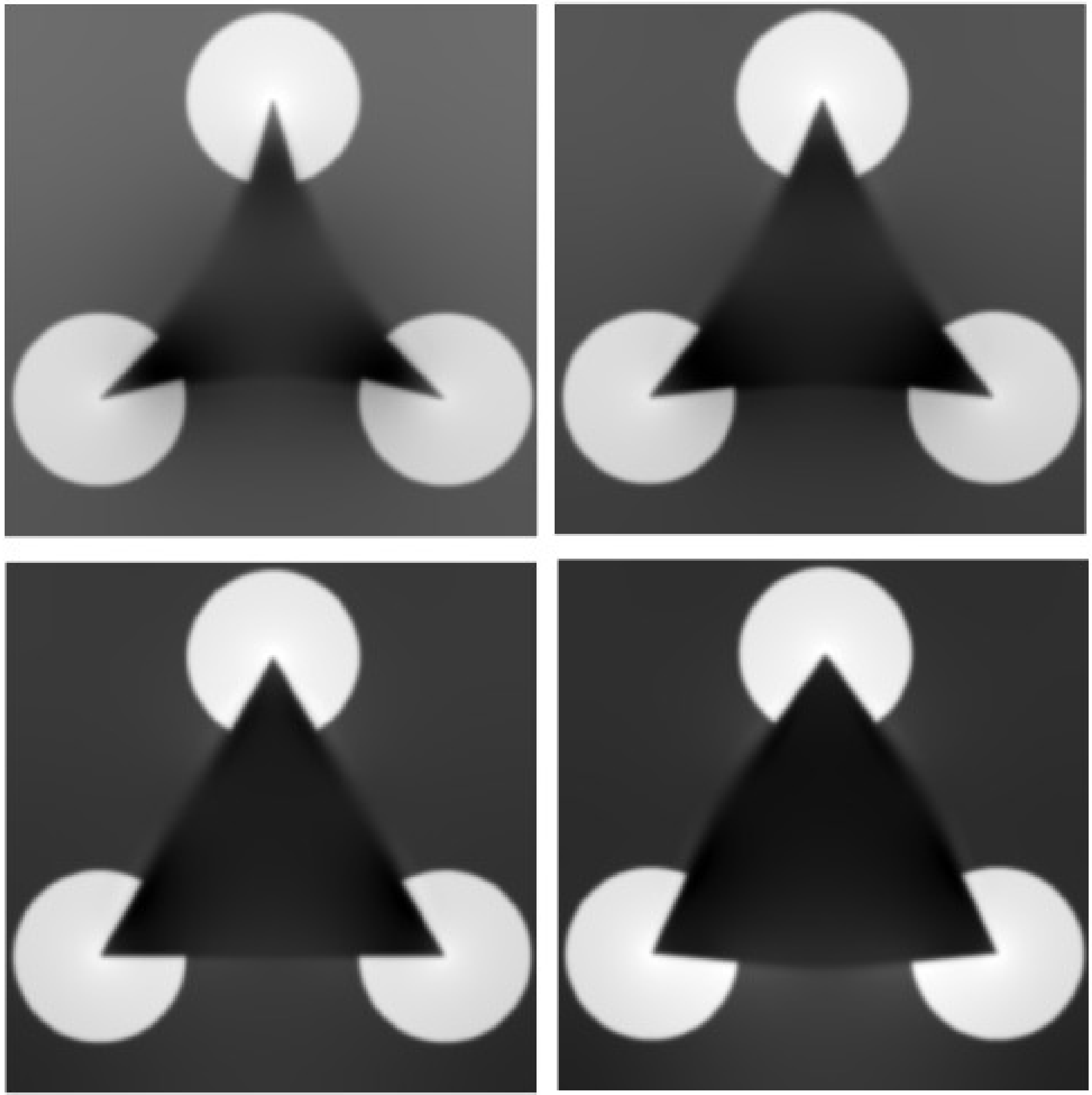
$$\Delta\phi = \frac{1}{2}(\Delta h + div(\vec{A}))$$



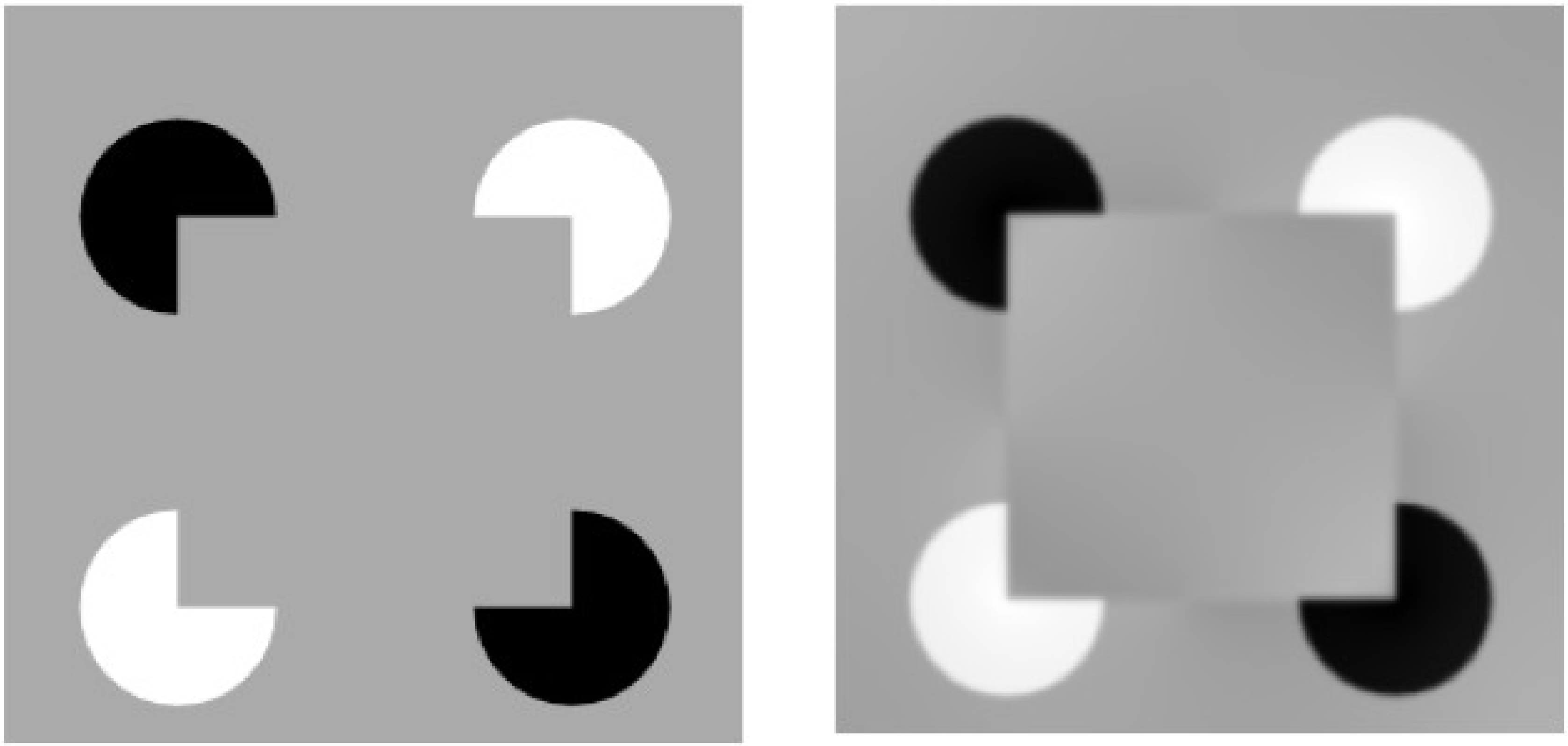
G.Citti, A.S. 2014

Inverted contrast

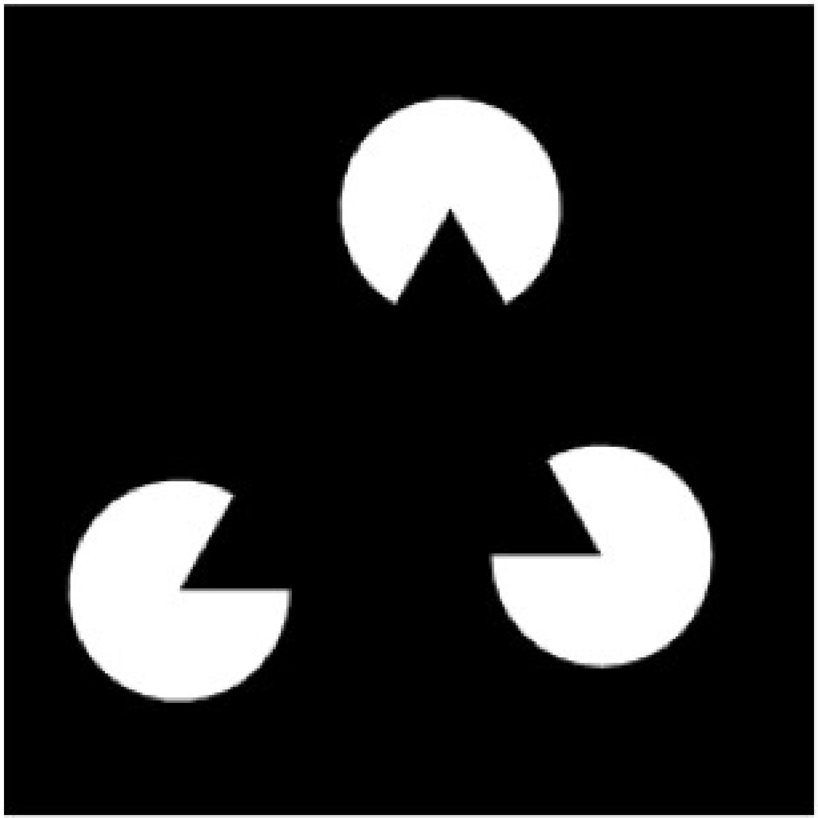




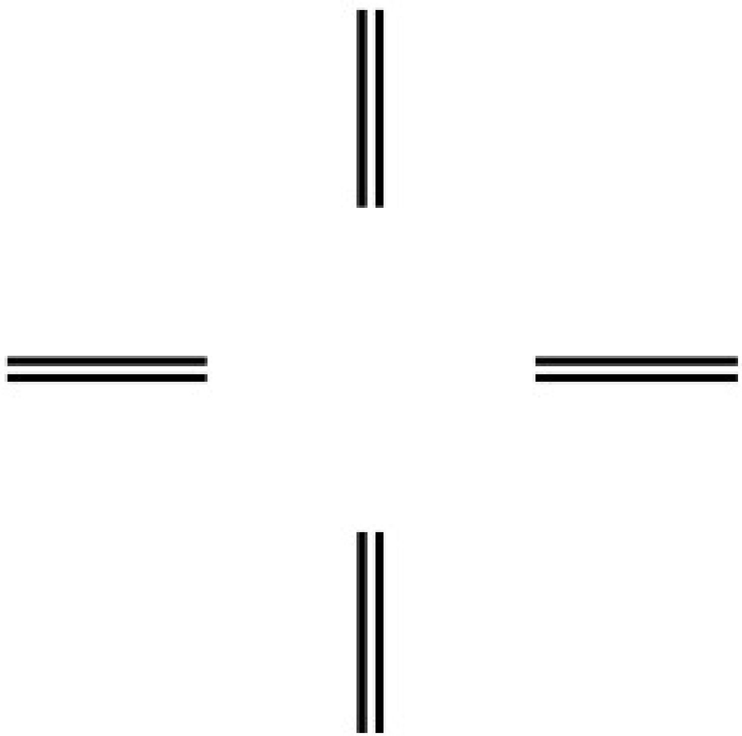
Alternate polarity

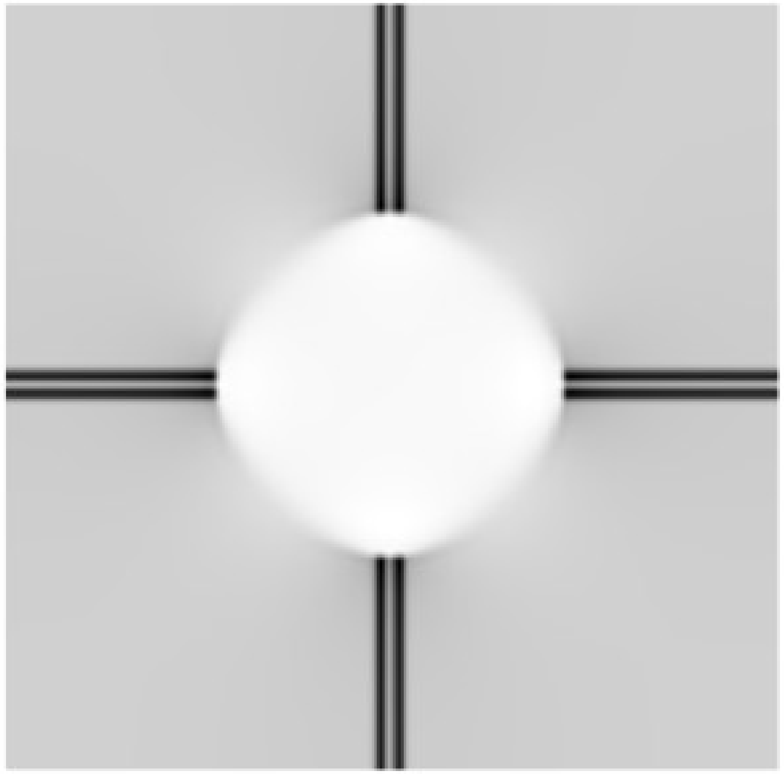


Fragmentation

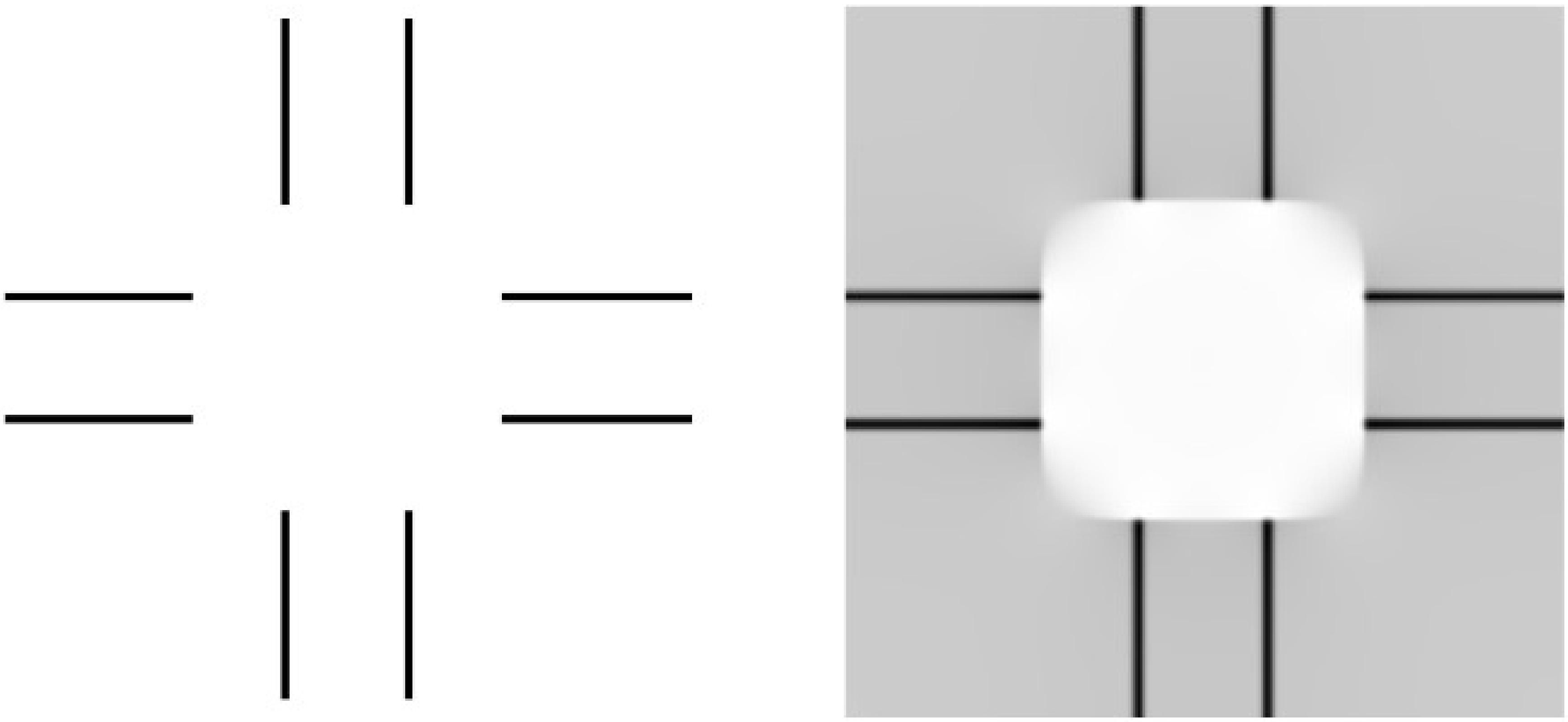


Koffka cross: narrow





Koffka cross: wide

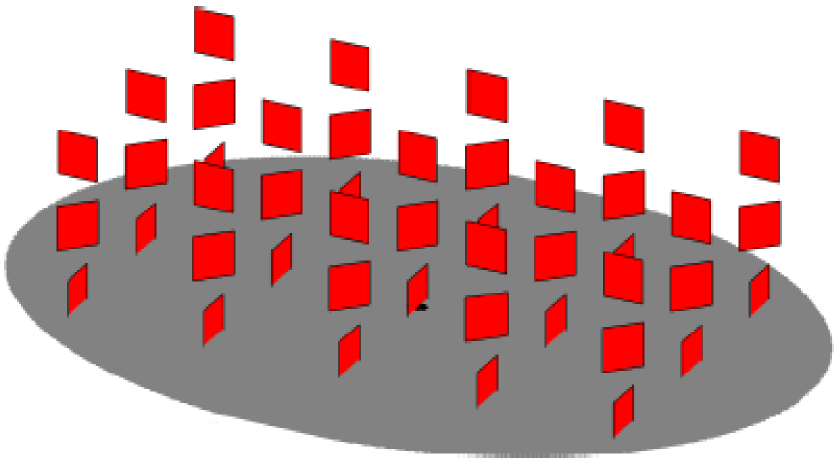


Cortico-cortical heterogeneity

Lifting of Xi in B0

$$X_{i,p} \quad p \in B_{p_0}$$

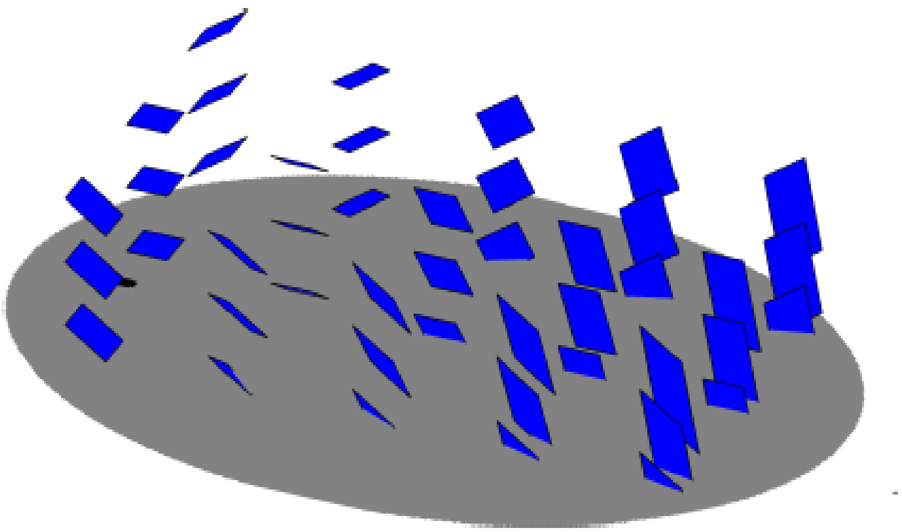
$$[X_i, X_j]$$



Lifting of Yi in B1

$$Y_{i,p} \quad p \in B_{p_1}$$

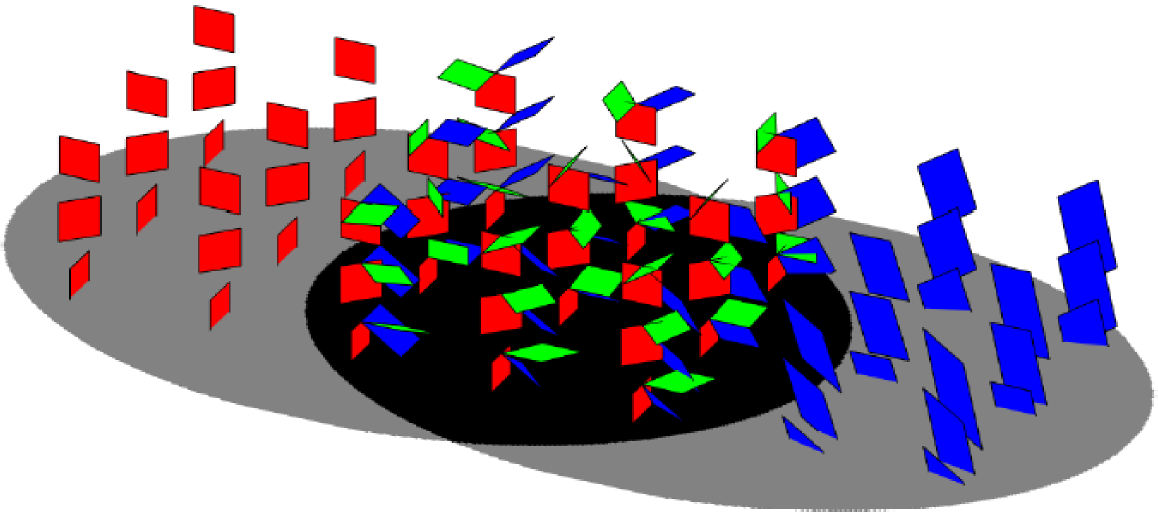
$$[Y_i, Y_j]$$



Cortico-cortical heterogeneity
Lifting in the intersection

$B_{p_0} \cap B_{p_1}$

$$X_{i,p}$$
$$Y_{i,p}$$
$$[X_i, X_j]$$
$$[Y_i, Y_j]$$
$$[X_i, Y_j]$$



L.P.Rotschield, E.M.Stein, 1976
A.S, G.Citti, D.Piotrowski, 2019