Quotients of $\mathbb{R}^8$ & Reduced Holonomy

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In honour of Dmitri Alekseevsky

9 September 2020

Topics to interest Dmitri

...contained in this talk include

– Riemannian holonomy groups
– Einstein metrics
– Homogeneous spaces
– Quaternionic geometry
– Twistor spaces

Relevant papers

[ABS] B. Acharya, R. Bryant, S. Salamon: in DGA 2020

[KL] S. Karigiannis, J. Lotay: arXiv:2002.06444

[FHN] L. Foscolo, M. Haskins, J. Nordstrom: arXiv:1805.02612

[K] K. Kawai: in Comm. Anal. Geom. 2018

[B] O. Bogoyavlenskaya: in Sibirsk. Mat. Zh. 2013

[AS] V. Apostolov, S. Salamon: in Comm. Math. Phys. 2004

[AyW] M. Atiyah, E. Witten: in Adv. Theor. Math. Phys. 2002

[AW] B. Acharya, E. Witten: arXiv:hep-th/0109152

Today's talk

If N^6 is nearly Kähler then the cone $\mathbb{R}^+ \times N$ has a Ricci-flat metric with holonomy in G_2.

We shall take $N = \mathbb{CP}^3$ with its NK structure and non-integrable almost complex structure J_2 arising from the fibration $\mathbb{CP}^3 \to S^4$. Set $\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$. We shall

- construct the resulting G_2 metric h on $\mathscr{C}$ starting from $\mathbb{C}^4 = \mathbb{R}^8$,
- investigate the geometry arising from an action of $\mathrm{SO}(2)$ rotating S^4,
- explain that the quotient $\mathscr{C}/\mathrm{SO}(2)$ is essentially $\mathbb{R}^6$,
- describe the induced $\mathrm{SU}(3)$ structure $(\sigma, g, \mathbb{J})$ on $\mathbb{R}^6$ in the spirit of [AS].

The metric h can be smoothed into a complete asymptotically conical (AC) metric on the total space of $\Lambda^2_- T^*S^4$ [BS]. There are analogous AC metrics formed from the NK spaces $\mathrm{SU}(3)/T^2$ and $S^3 \times S^3$, though the last one is the most amenable for study (next slides).

Motivation from ALC families

The AC metric on the spin bundle over S^3 with isometry group $\mathrm{SU}(2)^3$ represents a bifurcation in a one-parameter family of G_2 metrics with a cohomogeneous-one action by $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$, in two different ways giving a G_2 flop [FHN].

The AC metric is a limit of asymptotically locally conical (ALC) metrics, each of which has a circle of fixed radius r at infinity. These ALC metrics first appeared in the physics literature [BGGG, CGLP,...] with the names $\mathbb{B}_7$ and $\mathbb{D}_7$ and the existence of one was proved by [B]. In the collapsed limit as $r \to 0$, one obtains an AC Calabi-Yau space.

Circle bundles over singular Calabi-Yau spaces can be used to construct G_2 metrics [FHN′]. There is an infinite family of complete AC G_2 metrics on circle bundles $M_{m,n} \to K_{\mathbb{CP}^1\times\mathbb{CP}^1}$ that are asymptotic to cones over finite quotients of $S^3 \times S^3$.

G_2 metrics of cohomogeneity one

...based on one-parameter families of half-flat $SU(3)$ structures on $S^3 \times S^3$ invariant by $SU(2)^2 \times U(1)$:

$S^3 \times \Lambda^2_-$

collapsed

incomplete regime

$C(S^3 \times S^3)$

$C(S^2 \times S^3)$

$S^2 \times \mathbb{R}^4$

Join the dots!

By analogy, the SO(5)-invariant G_2 metric on $\Lambda^2_- T^*S^4$ arises as a collapsed limit of metrics with holonomy Spin(7) on the spin bundle over S^4.

Dirac monopole: U(1) acts on the left on $\mathbb{H}$ with quotient $\boxed{\mathbb{R}^4/\mathrm{U}(1) \cong \Lambda^2_-}$

Let's return to $\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$ with its conical G_2 metric h. There is no obvious way to associate ALC metrics to this set-up because of the absence of free circle and group actions. Nonetheless, there is an action of SO(2) on

$$S^4 \subset \mathbb{R}^2 \oplus \mathbb{R}^3$$

that lifts to $\mathbb{CP}^3$ and fixes two 2-spheres. Then $\mathscr{C}/\mathrm{SO}(2)$ has singular locus $\mathbb{R}^3 \cup \mathbb{R}^3$ (minus the origin). M-theory formulated on $\mathscr{C}$ is dual to Type IIA superstring theory on $\mathbb{R}^6$, and fixed points of SO(2) on $\mathscr{C}$ are identified with D-branes of the quotient [AyW].

Group actions

Start with $\mathbb{H}^2 = \mathbb{C}^4 = \mathbb{R}^8$. Its QK structure corresponds to $\mathrm{Sp}(2)_\ell \times \mathrm{Sp}(1)_r$ modulo $\mathbb{Z}_2$. Consider the subgroups

$$\mathrm{U}(1)_\ell \times \mathrm{SU}(2) \subset \mathrm{Sp}(2)_\ell, \qquad \mathrm{Sp}(1)_r \supset \mathrm{U}(1)_r.$$

The 2-torus $\mathrm{U}(1)_\ell \times \mathrm{U}(1)_r$ acts on $\mathbb{H}^2 = \mathbb{C}^4$ as

$$\begin{aligned} (q_0, q_1) &\mapsto e^{i\theta}(q_0, q_2)e^{i\phi} \\ (z_0, z_1, z_2, z_3) &\mapsto (e^{i(\theta+\phi)}z_0, e^{i(\theta-\phi)}z_1, e^{i(\theta+\phi)}z_2, e^{i(\theta-\phi)}z_3). \end{aligned}$$

It splits $\mathbb{R}^8 = \mathbb{R}^4_{0145} \oplus \mathbb{R}^4_{2367}$ with a ‘transposed’ hyperkähler structure associated to

$$\Lambda^2_-(\mathbb{R}^4_{0145}) \oplus \Lambda^2_-(\mathbb{R}^4_{2367}) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathbb{R}^6.$$

This space is T^2 invariant, and is the target of an associated moment map.

Define $\mathbb{CP}^3 = S^7/\mathrm{U}(1)_r$ and set $\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$. Then the moment map induces

$$\begin{array}{rccc} \mu: & \mathscr{C} & \longrightarrow & \mathbb{R}^6 \\ & [z_0, z_1, z_2, z_3] & \longmapsto & (\mathbf{u}, \mathbf{v}), \end{array}$$

whose fibres are orbits of $\mathrm{U}(1)_\ell/\mathbb{Z}_2 = \mathrm{SO}(2)$ à la Gibbons-Hawking, and

$$\left\{ \begin{array}{rcl} u_1 & = & |z_0|^2 - |z_2|^2 \\ u_2 & = & 2\,\mathrm{Re}(z_0\bar{z}_2) \\ u_3 & = & -2\,\mathrm{Im}(z_0\bar{z}_2), \end{array} \right. \qquad \left\{ \begin{array}{rcl} v_1 & = & |z_1|^2 - |z_3|^2 \\ v_2 & = & 2\,\mathrm{Re}(z_1\bar{z}_3) \\ v_3 & = & 2\,\mathrm{Im}(z_1\bar{z}_3). \end{array} \right.$$

$$R = \sum_{i=0}^{3} |z_i|^2 \text{ equals } u + v, \text{ where } u = |\mathbf{u}| \text{ and } v = |\mathbf{v}|.$$

The action of T^2 on $\mathbb{R}^8$ commutes with $\mathrm{SU}(2)$ that acts as $\mathrm{SO}(3)$ diagonally on $\mathbb{R}^6$.

Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a triple of 1-forms on $\mathbb{R}^8$ trivializing the action of $\mathrm{Sp}(1)_r$, chosen so that $\alpha_1^\sharp$ generates $\mathrm{U}(1)_r$.

Proposition. The G$_2$ 3-form φ on $\mathscr{C}$ equals $-d\big(R\tau\big)$, where

$$\tau = dR \wedge \alpha_1 - \alpha_2 \wedge \alpha_3.$$

To smooth the vertex ($r = 0$) of the cone, replace the coefficient R of τ by $(R^4 + 1)^{1/4}$. The resulting complete AC metric is

$$(R^4 + 1)^{-1/2} g_{\mathrm{ver}} + (R^4 + 1)^{1/2} g_{\mathrm{hor}}.$$

It has convergence rate -4 (since R is Euclidean radius squared) and is rigid as an AC metric [KL′].

Parallel 3-vectors

We are now considering the quotient of the G_2 manifold $\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$ by $SO(2)$, which is $\mathbb{R}^6 \setminus \mathbf{0}$. The $SO(3)$ orbit of a bivector $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^6$ has dimension 3 unless $\mathbf{u} \wedge \mathbf{v} = 0$.

Definition. Set $\mathscr{F}_\pm = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^6 : v\mathbf{u} = \pm u\mathbf{v}\}$.

The equation $uv = 0$ defines the singular locus $\mathbb{R}^3 \cup \mathbb{R}^3$ of $\mathbb{R}^6$ where the circle fibres of μ collapse. If $uv \neq 0$ then $(\mathbf{u}, \mathbf{v}) \in \mathscr{F}_+$ (resp. $\mathscr{F}_-$) iff $\mathbf{u}, \mathbf{v}$ are aligned (resp. anti-aligned).

We can interpret these sets in terms of the fibration $\mathbb{CP}^3 \to S^4$ (next slide):

Theorem. $\mu^{-1}(\mathscr{F}_\pm)/\mathbb{R}^* = Q_\pm \subset \mathbb{CP}^3$ where

$Q_+ = \{[z_0, z_1, z_2, z_3] : z_0\bar{z}_3 - z_1\bar{z}_2 = 0\}$ consists of points where $U(1)_\ell$ acts vertically,

$Q_- = \{[z_0, z_1, z_2, z_3] : z_0 z_1 + z_2 z_3 = 0\}$ consists of points where $U(1)_\ell$ acts horizontally.

The twistor fibration

Rather than using the Hopf map $\mathbb{CP}^3 \to \mathbb{HP}^1$, one can pass directly to the 4-sphere:

$$\begin{array}{ccccc} \boxed{\begin{array}{c}\mathbb{C}^4 \\ \downarrow \\ \Lambda^2_0(\mathbb{C}^4)\end{array}} & & & \begin{array}{c}\mathbb{CP}^3 \times \mathbb{R}^+ \quad = \mathscr{C} \\ \pi\downarrow \\ \end{array} \\ \cong \mathbb{R}^5 & \supset & S^4 \end{array}$$

The action of $\mathrm{U}(1)_\ell$ on $\mathbb{R}^8$ covers a rotation of S^4:

$$\begin{array}{ccccc} \mathrm{U}(1)_\ell & \subset & \mathrm{U}(2) & \subset & \mathrm{Sp}(2)_\ell \\ & & & & \downarrow \\ \mathrm{SO}(2) & \subset & \mathrm{SO}(2)\times\mathrm{SO}(3) & \subset & \mathrm{SO}(5). \end{array}$$

Let $\mathbb{S}^1 = S^4 \cap \mathbb{R}^2$ denote the fixed point set for the action of $\mathrm{SO}(3)$ ($s = 0$ next),
Let $\mathbb{S}^2 = S^4 \cap \mathbb{R}^3$ denote the fixed point set for the action of $\mathrm{SO}(2)$ ($s = 1$ next).

The two quadrics

The non-holomorphic quadric Q_+ is simply $\pi^{-1}(\mathbb{S}^2) \cong S^2 \times S^2$.

By contrast, Q_- contains $\pi^{-1}(\mathbb{S}^1)$, away from which it is a double covering of

$$S^4 \setminus \mathbb{S}^1 \;\cong\; \mathbb{H} \setminus \mathbb{R} \;\cong\; S^2 \times \mathcal{H}^2.$$

It encodes the conformally Kähler metric [Pont,SV] and the orthogonl complex structure on $\mathbb{H} \setminus \mathbb{R}$ that can be used to define quaternion power series [GSS].

If X is the Killing field generated by $\mathrm{SO}(2)$, then

$$X^\flat = (1 - s^2)dt,$$

where $t\colon S^4 \setminus \mathbb{S}^2 \to [0, 2\pi)$ and $s\colon S^4 \to [0, 1]$.

Circle actions (in general)

The 'dual pair' $\mathrm{SO}(2) \times \mathrm{SO}(3)$ (arising from $\mathrm{U}(2) \subset \mathrm{Sp}(2)_\ell$) acts on $\mathbb{CP}^3$ and $\mathscr{C}$. We have already parametrized the orbits of $\mathrm{SO}(2)$, and will deal with those of $\mathrm{SO}(3)$ shortly.

One could instead focus on

$$\mathrm{U}(1) \times \mathrm{Sp}(1) \subset \mathrm{Sp}(2)_\ell$$

that acts as $\mathrm{U}(2)$ on $\mathbb{R}^4$ fixing two poles of S^4. Or work with arbitrary weights for the action of a circle subgroup of $\mathrm{U}(2)$ on $\mathbb{C}^2$.

Backtracking, we could replace $\mathrm{U}(1)_r$ by $\mathrm{U}(1)$ with weights (p, q) on $\mathbb{H}^2$ giving rise to weighted projective space $\mathbb{WCP}^3_{p,p,q,q}$ with a circle action again fixing two projective lines. This space is conjectured to carry a NK metric [AW].

3-dimensional orbits

Let G be a compact Lie group, for instance $\mathrm{SO}(5)$.

Key fact. Each conjugacy class of subalgebras $\mathfrak{su}(2) \subset \mathfrak{g}$ gives rise to a complex nilpotent orbit $\mathscr{N} \subset \mathfrak{g}_{\mathbb{C}}$ with a HK metric [Kr], and a (typically incomplete) QK metric on the total space M^{4n} of a vector bundle over $L = G/N(\mathfrak{su}(2))$ [Sw].

There are 3 such classes for $\mathfrak{so}(5)$:

- the minimal $\mathfrak{su}(2)$ with normalizer $\mathrm{SO}(4)$, so $L = M = S^4$ and $n = 1$
- our $\mathfrak{so}(3) = \mathfrak{su}(2)$ with $L = SO(5)/SO(2)\times SO(3) \cong Q^3$ and $n = 2$
- the principal $\mathfrak{su}(2)$ with $L = SO(5)/SO(3)$ and $n = 3$.

In the last case, C has a nearly-parallel G_2 structure, so $\mathbb{R}^+ \times C$ has holonomy $\mathrm{Spin}(7)$.

Coassociative submanifolds

Work inside the 7-manifold $\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$, with its 3-form φ defining the G_2 metric h.

Let $i\colon \mathcal{U} \to \mathscr{C}$ be a 3-dimensional orbit of some subgroup SU(2) or SO(3) of SO(5). Since the latter preserves φ, $i^*\varphi$ must be a constant multiple of the volume form on $\mathcal{U}$. But $[i^*\varphi] = i^*[-d(R\tau)] = 0$, so $i^*\varphi \equiv 0$.

Lemma. A 4-dimensional submanifold $\mathcal{V}$ of a G_2 manifold is coassociative iff $i^*\varphi \equiv 0$.

We expect each orbit $\mathcal{U}$ to be contained in a unique such a submanifold, $T_x\mathcal{U} \subset T_x\mathcal{V}$.

In favourable circumstances, there will be a foliation of $\mathscr{C}$ by coassociatives of codim one [K, KL]. ($\mathscr{C}$ has a more elementary foliation by Eguchi-Hanson spaces T^*S^2, each with $\mathbf{u} - \mathbf{v}$ constant, but this is not really related to G_2.)

Invariants for our SO(3)

For the diagonal action of SO(3) on $\mathbb{R}^6$, invariant functions are obviously u, v and $\mathbf{u}\cdot\mathbf{v}$. We can recover these on $\mathbb{CP}^3$ from the function $z_0\bar{z}_3 - \bar{z}_1 z_2 = \sqrt{a/2}\,e^{it}$ defining Q_+:

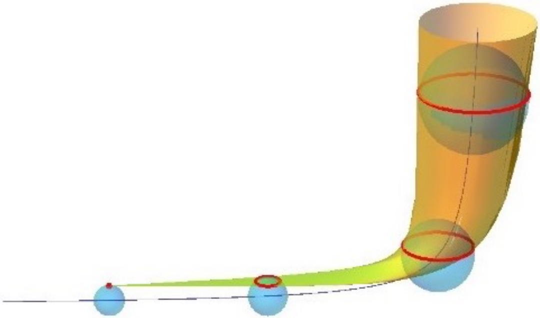
Lemma. $2a = uv - \mathbf{u}\cdot\mathbf{v} = R^2(1-s^2)$, so $R = \sqrt{\dfrac{2a}{1-s^2}}$.

An SO(3) orbit $\mathcal{U}$ will intersect a twistor fibre S^2 of radius R over $p \in S^4 \setminus \mathbb{S}^1$ in a circle at 'height' $h = (u-v)/(Rs) \in [-1,1]$ relative to poles defined by Q_-.

Define $b = u^2 - v^2 = RHs$, so

$$h = (b/a^2)\frac{1-s^2}{s}.$$

Theorem [ABS]. Setting a, b, t constant defines a coassociative submanifold of $\mathscr{C}$ diffeomorphic to T^*S^2 unless $a = b = 0$.



Symplectic form σ

This is induced from the G_2 structure on $\mathscr{C}$. Let X be the Killing vector field generating the SO(2) fibres of $\mu\colon \mathscr{C} \to \mathbb{R}^6$. Then

$$\sigma = X \lrcorner\, \varphi = -X \lrcorner\, d(R\tau) = d(RX \lrcorner\, \tau).$$

Set $\mathbf{p} = \mathbf{u} + \mathbf{v}$ and $\mathbf{q} = R(\mathbf{u} - \mathbf{v})$, where $R = u + v = |\mathbf{u}| + |\mathbf{v}|$.

Unexplained theorem [ABS]. The components of $\mathbf{p}, \mathbf{q}$ are Darboux coordinates:

$$\sigma = -\tfrac{1}{2}\sum_{i=1}^{3} dp_i \wedge dq_i.$$

Note that σ extends to $\mathbb{R}^3 \cup \mathbb{R}^3$ and is non-degenerate on $\mathbb{R}^6 \setminus \mathbf{0}$. The projections $(\mathbf{u}, \mathbf{v}) \mapsto R^{1/2}\mathbf{u}$ and $(\mathbf{u}, \mathbf{v}) \mapsto R^{1/2}\mathbf{v}$ also have Lagrangian fibres.

Induced metric g

Recall that h is the conical metric on $\mathscr{C}$ with holonomy G_2. We seek the metric g induced on $\mathbb{R}^6 \setminus (\mathbb{R}^3 \cup \mathbb{R}^3)$ by setting

$$h = \mu^* g + \tfrac{1}{4} N \Theta^2,$$

where $\Theta = 2(X \lrcorner\, h)/N$ is the connection 1-form, and $N = h(X,X) = 6uv - 2\mathbf{u}\cdot\mathbf{v}$ measures the size of the circle fibres. This makes μ a Riemannian submersion.

Computational theorem [ABS].

$$g = \tfrac{1}{2} dR^2 + \tfrac{1}{2}\left|d\mathbf{u} + d\mathbf{v}\right|^2 + \tfrac{2}{N}\left|u\,d\mathbf{v} - v\,d\mathbf{u}\right|^2 + \tfrac{1}{2N}\Gamma_+^2 - \tfrac{1}{4N}\Gamma_-^2,$$

where the 1-forms

$$\begin{aligned} \Gamma_+ &= \mathbf{u}\cdot d\mathbf{v} + \mathbf{v}\cdot d\mathbf{u} - u\,dv - v\,du, \\ \Gamma_- &= \mathbf{u}\cdot d\mathbf{v} - \mathbf{v}\cdot d\mathbf{u} + u\,dv - v\,du. \end{aligned}$$

vanishes on $\mathscr{F}_+$, $\mathscr{F}_-$ respectively.

Non-degeneracy

We can recover the conical nature of g by the change of variables

$$u = R\cos^2(\phi/2), \qquad v = R\sin^2(\phi/2).$$

Then $\mathscr{F}_\pm \cong \mathbb{R}^+ \times [0,\pi] \times S^2$, with coordinates $R, \phi, \boldsymbol{\sigma}$.

Corollary 1. The restriction of g to $\mathscr{F}_\pm$ equals $dR^2 + R^2\widehat{g}$ where

$$\widehat{g} = \begin{cases} \frac{1}{2}|d\boldsymbol{\sigma}|^2 + \frac{1}{4}d\phi^2 & \text{on } \mathscr{F}_+ \\ \frac{1}{8}(3+\cos 2\phi)|d\boldsymbol{\sigma}|^2 + \frac{1}{2}d\phi^2 & \text{on } \mathscr{F}_-. \end{cases}$$

Corollary 2. Relative to g, vectors in the respective singular $\mathbb{R}^3$ axes meet at an angle of

$$\tfrac{1}{2}\pi \leqslant \pi\sqrt{\tfrac{3}{8} - \tfrac{1}{8}\cos\theta} \leqslant \tfrac{1}{\sqrt{2}}\pi \sim 127^{\text{o}}.$$

This is determined by g and the closed 3-form $\psi^+ = X \lrcorner\, (*\varphi)$.

Proposition.
$$8uv\psi^+ = \tfrac{1}{3}v(2v^2 + 3uv - \mathbf{u}\cdot\mathbf{v})\{d\mathbf{u}, d\mathbf{u}, d\mathbf{u}\}$$
$$-v(4u^2 + 3uv + \mathbf{u}\cdot\mathbf{v})\{d\mathbf{v}, d\mathbf{u}, d\mathbf{u}\} + \big((u+2v)\mathbf{v}\cdot d\mathbf{v} + v\mathbf{u}\cdot d\mathbf{v}\big) \wedge \{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\}$$
$$+(v\mathbf{u}\cdot d\mathbf{v} - u\mathbf{v}\cdot d\mathbf{v}) \wedge \{\mathbf{v}, d\mathbf{u}, d\mathbf{u}\} + \text{ terms interchanging } \mathbf{u} \text{ and } \mathbf{v}.$$

Let $\mathbf{n} \in S^2$ and set $\mathscr{M}_\mathbf{n} = \{(\mathbf{u},\mathbf{v}) : \mathbf{u}\cdot\mathbf{n} = 0 = \mathbf{v}\cdot\mathbf{n},\ uv \neq 0\}$, essentially an $\mathbb{R}^4$.
Note that $\mathscr{M}_\mathbf{n} = \mathscr{M}_{\mathbf{n}'}$ iff $\mathbf{n} = \pm\mathbf{n}'$, otherwise the intersection is 2-dimensional.

Corollary 3. $\mathscr{M}_\mathbf{n}$ is a non-integrable $\mathbb{J}$-holomorphic subvariety of $\mathbb{R}^6$.

An open subset of $\mathbb{R}^6$ is exhausted by a family of $\mathbb{J}$-holomorphic surfaces parametrized by $\mathbb{RP}^2$. The intersection of any two is a ‘superminimal’ $\mathbb{J}$-holomorphic curve.

Upcoming talks

– S.-T. Yau, Simons Foundation lecture, tonight!

– T. Madsen, Multisymplectic Geometry Leuven, in 24 hours.

– K. Dixon, SCSHGAP workshop, in 7 days.

With admiration