The Lichnerowicz Laplacian and Prescribed Ricci problem on homogeneous manifolds

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Figure: Alesund, Norway, June 2019.

Some old open questions

- Is the bi-invariant metric on $\mathrm{SU}(3)$ isolated among Einstein left-invariant metrics?
- Does $\mathrm{SU}(3)$ admit a third Einstein left-invariant metric? Are there only finitely many up to isometry and scaling?
- Same questions for most compact homogeneous spaces.
- Does $\mathrm{SL}_2(\mathbb{C})$ admit an Einstein left-invariant metric?
- Same question for any non-compact simple Lie group other than $\mathrm{SL}_2(\mathbb{R})$. Expected answer is NO, according to the Alekseevsky Conjecture (i.e., Einstein solvmanifolds exhausts the possibilities for Einstein homogeneous of negative scalar curvature).

Setting

M homogeneous manifold,

G connected unimodular Lie group acting transitively on M,

$\mathcal{M}^G$ set of G-invariant metrics on $M = G/K$, K isotropy at $o \in M$, almost-effective,

$\mathcal{M}^G$ is an open cone of the finite-dimensional vector space $\mathcal{S}^2(M)^G$ of G-invariant symmetric 2-forms,

$$1 \leq \dim \mathcal{M}^G \leq \frac{n(n+1)}{2}.$$

The manifold $\mathcal{M}^G$ can be naturally endowed with a Riemannian metric:

$$\langle T, T \rangle_g := \sum T(X_i, X_i)^2,$$

where $\{X_i\}$ is any g_o-orthonormal basis of T_oM. Consider,

$$\mathrm{Rc} : \mathcal{M}^G \to \mathcal{S}^2(M)^G, \qquad \mathrm{Sc} : \mathcal{M}^G \to \mathbb{R}.$$

Variational approach to Einstein metrics

It is well known ([Wang-Ziller, Nikonorov, Heber]) that

$$\mathrm{Sc} : \mathcal{M}^G \to \mathbb{R}, \qquad \mathrm{grad}(\mathrm{Sc})_g = -\mathrm{Rc}(g), \qquad \forall g \in \mathcal{M}^G.$$

If $\mathcal{M}_1^G := \{g \in \mathcal{M}^G : \det_{\overline{g}} g = 1\}$, then $T_g\mathcal{M}_1^G = (\mathbb{R}g)^{\perp_g}$.

Lemma ([Palais 70])

$g \in \mathcal{M}_1^G$ is a critical point of $\mathrm{Sc}|_{\mathcal{M}_1^G}$ if and only if g is Einstein (i.e., $\mathrm{Rc}(g) = \rho g$, $\rho \in \mathbb{R}$).

On the other hand, the Einstein operator (or traceless Ricci tensor) is defined by

$$\mathrm{E} : \mathcal{M}^G \longrightarrow \mathcal{S}^2(M)^G, \qquad \mathrm{E}(g) := \mathrm{Rc}(g) - \tfrac{\mathrm{Sc}(g)}{n} g,$$

so g is Einstein if and only if $\mathrm{E}(g) = 0$.

Stability and non-deformability towards rigidity

$\mathrm{Aut}(G/K) \circlearrowleft \mathcal{M}^G$ giving rise to equivariant isometry classes.

$$T_g\mathcal{M}^G = \mathcal{S}^2(M)^G = (\mathbb{R}g \oplus T_g \,\mathrm{Aut}(G/K)\cdot g) \oplus^{\perp_g} W_g.$$

Definition

An Einstein metric $g \in \mathcal{M}^G$ is said to be,

- stable: $\mathrm{Sc}''_g|_{W_g\times W_g} < 0$, where $\mathrm{Sc}''_g(T,T) := \frac{d^2}{dt^2}\Big|_0 \mathrm{Sc}(g+tT)$.
- unstable: $\mathrm{Sc}''_g(T,T) > 0$ for some $T \in \mathcal{S}^2(M)^G$.
- infinitesimally deformable: $\mathrm{Ker}\, d\,\mathrm{E}|_g \cap W_g \neq 0 \quad (\mathrm{E}(g) = \mathrm{Rc}(g) - \frac{\mathrm{Sc}(g)}{n} g)$.
- infinitesimally non-deformable: $\mathrm{Ker}\, d\,\mathrm{E}|_g \cap W_g = 0$
- rigid: $\exists$ open neighborhood U of g in $\mathcal{M}^G$ s.t. any other Einstein $g' \in U$ belongs to $\mathbb{R}_+ \mathrm{Aut}(G/K)\cdot g$ ($\Rightarrow$ isolated if $\mathrm{Aut}(G/K) \subset \mathrm{Iso}(M,g)$).

[Koiso 80] stable $\Rightarrow$ infinitesimally non-deformable $\Rightarrow$ rigid.

$$\mathrm{Rc} : \mathcal{M}^G \to \mathcal{S}^2(M)^G, \qquad d\,\mathrm{Rc}|_g : \mathcal{S}^2(M)^G \to \mathcal{S}^2(M)^G.$$

Lemma ([E. Lauret-L. 20])

For any Einstein $g \in \mathcal{M}^G$, say $\mathrm{Rc}(g) = \rho g$,

- $\mathrm{Sc}''_g(T,T) = 4\langle(\rho\,\mathrm{id} - d\,\mathrm{Rc}|_g)T, T\rangle_g$, *for any* $T \in \mathcal{S}^2(M)^G$.
- $d\,\mathrm{E}|_g = d\,\mathrm{Rc}|_g - \rho\,\mathrm{id}$ *on* $(\mathbb{R}g)^{\perp_g}$.

Corollary

An Einstein $g \in \mathcal{M}^G$, say $\mathrm{Rc}(g) = \rho g$, is

- *stable if and only if* $\mathrm{Spec}\left(d\,\mathrm{Rc}|_g|_{W_g}\right) > \rho$;
- *unstable if and only if* $d\,\mathrm{Rc}|_g|_{W_g}$ *has an eigenvalue* $< \rho$;
- *infinitesimally non-deformable if and only if* $\rho \notin \mathrm{Spec}\left(d\,\mathrm{Rc}|_g|_{W_g}\right)$;
- *infinitesimally deformable if and only if* $\rho \in \mathrm{Spec}\left(d\,\mathrm{Rc}|_g|_{W_g}\right)$.

Prescribed Ricci problem [Besse, Chapter 5]. Given $T \in \mathcal{S}^2(M)^G$, existence and uniqueness of $g \in \mathcal{M}^G$ and a constant $c > 0$ such that

$$\text{(PRP)} \qquad \boxed{\operatorname{Rc}(g) = cT}$$

- Existence in (PRP) $\Leftrightarrow$ image of $\operatorname{Rc} : \mathcal{M}^G \longrightarrow \mathcal{S}^2(M)^G$ up to scaling.
- $\dim \operatorname{Ker} d \operatorname{Rc}|_g$ is relevant (always $\mathbb{R}g \subset \operatorname{Ker} d \operatorname{Rc}|_g$).
- If $M = G_1/K_1 \times G_2/K_2$, $G = G_1 \times G_2$ and $g = g_1 + g_2 \in \mathcal{M}^G$, where g_i is a G_i-invariant metric on $M_i = G_i/K_i$, then $c_1 g_1 + c_2 g_2 \in \mathcal{M}^G$ and

  $$\operatorname{Rc}(c_1 g_1 + c_2 g_2) = \operatorname{Rc}(g_1) + \operatorname{Rc}(g_2) = \operatorname{Rc}(g), \qquad \forall c_1, c_2 > 0,$$

  giving rise to a non-uniqueness situation.
- $\dim \operatorname{Ker} d \operatorname{Rc}|_g \geq 2$ for any product metric $g = g_1 + g_2$ as above.
- If the isotropy representation of G_1/K_1 does not contain any trivial subrepresentation, then any $g \in \mathcal{M}^G$ is a product metric.

Some known results

Sufficient and/or necessary conditions for the solvability of $\mathrm{Rc}(g) = cT$ on large classes of homogeneous spaces. Mostly for $T \geq 0$, uniqueness always holds.

- [Pulemotov 16, 18; Gould-Pulemotov 17] Generalized Wallach spaces, generalized flag manifolds and two irreducible isotropy summands.
- [Buttsworth-Pulemotov-Rubinstein-Ziller 18] Spheres and projective spaces. Ricci iteration.
- [Buttsworth 19] Unimodular 3-dimensional Lie groups. Non-uniqueness of g and c occurs.
- [Arroyo-Pulemotov-Ziller 20] D'Atri-Ziller metrics on compact simple Lie groups (i.e., K-invariant metrics on a compact simple Lie group H, $K \subset H$, naturally reductive).
- [Arroyo-Gould-Pulemotov 20] K-invariant metrics on a non-compact simple Lie group H, $K \subset H$ maximal compact subgroup (naturally reductive). Non-uniqueness of c.

$$(\mathrm{PRP}) \qquad \boxed{\mathrm{Rc}(g) = cT}$$

- Given T, what kind of set is

$$\{c > 0 : \text{there exists solution } g \text{ to (PRP)}\} = ??$$

Is it bounded below? Is it finite?

- [DeTurck-Koiso 84] If M compact, $T > 0$ and $\mathrm{Sec}(T) < \frac{1}{n-1}$, then there is no g such that $\mathrm{Rc}(g) = T$.
- Existence of solution $\Leftrightarrow$ $T \in \mathbb{R}_+ \mathrm{Rc}(\mathcal{M}^G)$ $\Leftrightarrow$ image of

$$\widetilde{\mathrm{Rc}} : \mathcal{M}^G \longrightarrow \mathcal{S}^2(M)^G, \qquad \widetilde{\mathrm{Rc}}(g) := (\det_{g_0} g)\, \mathrm{Rc}(g).$$

In particular, $\widetilde{\mathrm{Rc}}(ag) = a^n \widetilde{\mathrm{Rc}}(g)$ for any $a > 0$.

- Injectivity of $\widetilde{\mathrm{Rc}}$ $\Leftrightarrow$ uniqueness of solutions (i.e., (g_1, c_1) and (g_2, c_2) are solutions iff $g_2 \in \mathbb{R}_+ g_1$ and $c_2 = c_1$).

Definition

A metric $\overline{g} \in \mathcal{M}^G$ is said to be Ricci locally invertible if there exist open neighborhoods U and V of $\overline{g}$ and $\mathrm{Rc}(\overline{g})$ in $\mathcal{M}^G$ and $\mathcal{S}^2(M)^G$, respectively, satisfying the following properties:

(a) $\mathrm{Rc}(U)$ is a submanifold of codimension one in V and $\mathrm{Rc} : U_1 \to \mathrm{Rc}(U) = \mathrm{Rc}(U_1)$ is a diffeomorphism, where $U_1 := \{g \in U : \det_{g_0} g = \det_{g_0} \overline{g}\}$;

(b) for any $T \in V$, there exists a unique constant $c > 0$ such that $\mathrm{Rc}(g) = cT$ for some $g \in U$;

(c) g is the unique metric up to scaling in U such that $\mathrm{Rc}(g) = cT$.

- In other words, the function Rc is, locally, as surjective and injective as it can be.
- There may exist another constant $c' > 0$ such that $\mathrm{Rc}(g') = c'T$ for some $g' \notin U$.

Ricci local invertibility

The following metrics are known to be Ricci locally invertible:

- [Hamilton 84] The round metric on the n-sphere $\mathbb{S}^n$.
- [DeTurck 85] Irreducible symmetric spaces of compact type (Einstein metrics with $\operatorname{Ker}\Delta_L = \mathbb{R}g$ (e.g., irreducible and $\operatorname{Sec} \geq 0$)).
- [Delay 99, Delay-Herzlich 01] Real and complex hyperbolic spaces.

Ricci local invertibility

Definition

A metric $\overline{g} \in \mathcal{M}^G$ is said to be Ricci locally invertible if:

(a) $\operatorname{Rc} : U_1 \to \operatorname{Rc}(U_1)$ is a diffeomorphism;

(b) $\forall T \in V \; \exists$ unique $c > 0$ s.t. $\operatorname{Rc}(g) = cT$, $g \in U$;

(c) g unique metric up to scaling in U s.t. $\operatorname{Rc}(g) = cT$.

- The subset

$$\mathcal{M}^G_{inv} := \left\{ g \in \mathcal{M}^G : g \text{ is Ricci locally invertible} \right\}$$

  is open in $\mathcal{M}^G$ and $\operatorname{Rc}(\mathcal{M}^G_{inv})$ is open in $\operatorname{Rc}(\mathcal{M}^G)$. Are they dense?
- The concept of Ricci local invertibility is geometric, in the sense that $g \in \mathcal{M}^G_{inv}$ if and only if any $f^*g \in \mathcal{M}^G_{inv}$ for any $f \in \operatorname{Aut}(G/K)$.
- $d \operatorname{Rc}|_g : (\mathbb{R}g)^{\perp_g} \longrightarrow (\mathbb{R}g)^{\perp_g}$ is an isomorphism for any $g \in U_1$.

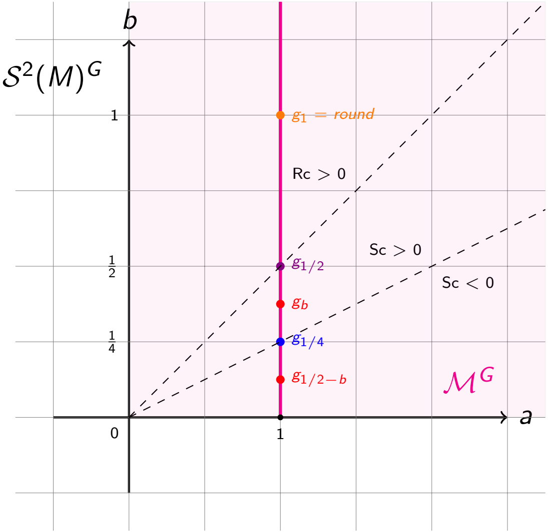

Figure: Berger spheres. $M = \mathbb{S}^3,\ G = \mathrm{SU}(2) \times S^1,\ g_b := (1, b, b) \in \mathcal{M}^G,\ b > 0.$

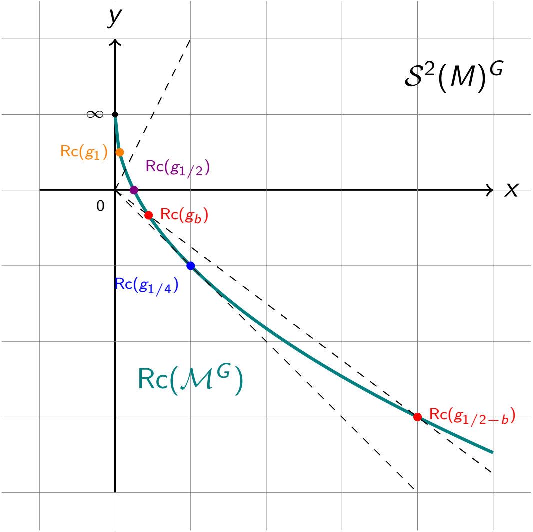

Figure: Ricci tensors of Berger spheres. $\mathrm{Rc}(g_b) = \left(\frac{1}{4b^2}, \frac{2b-1}{4b}, \frac{2b-1}{4b}\right)$.

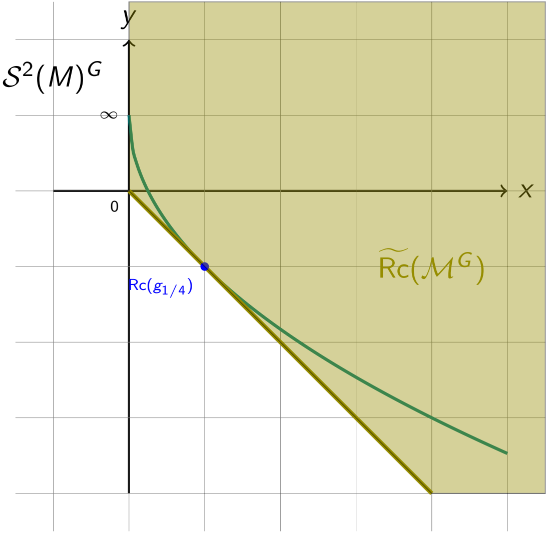

Figure: $\widetilde{\mathrm{Rc}} = \mathbb{R}_+ \, \mathrm{Rc}(\mathcal{M}^G)$ (i.e., solvable T's)

g Ricci locally invertible: the function Rc is, locally, as surjective and injective as it can be.

Theorem ([L.-Will 20])

The following conditions are equivalent for a metric $g \in \mathcal{M}^G$:

(i) $\widetilde{\mathrm{Rc}}$ *is a local diffeomorphism at g.*

(ii) $\mathrm{Ker}\, d\,\mathrm{Rc}\,|_g = \mathbb{R}g$ *and* $\mathrm{Sc}(g) \neq 0$.

In that case, g is Ricci locally invertible.

Corollary ([L.-Will 20])

The subset $\mathcal{M}^G_{inv}$ is either empty or open and dense in $\mathcal{M}^G$ (in particular, $\mathrm{Rc}(\mathcal{M}^G_{inv})$ is either empty or open and dense in $\mathrm{Rc}(\mathcal{M}^G)$).

- What is most likely? (recall the product case $M = G_1/K_1 \times G_2/K_2$).
- For G compact, $\widetilde{\mathrm{Rc}} : \mathcal{M}^G \to \mathcal{S}^2(M)^G$ is never a local diffeomorphism.

Moving bracket approach to compute $d\,\mathrm{Rc}|_g$ TIME!!

$M = G/K$, reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $T_oM \equiv \mathfrak{p}$,

$$\mathcal{S}^2(M)^G \leftrightarrow \mathrm{sym}^2(\mathfrak{p})^K, \qquad \mathcal{M}^G \leftrightarrow \mathrm{sym}^2_+(\mathfrak{p})^K.$$

Fix $g \in \mathcal{M}^G$, $\langle \cdot, \cdot \rangle := g_o$, μ Lie bracket of $\mathfrak{g}$, $\mu_{\mathfrak{p}} : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$.

The Ricci operator of g is given by

$$\mathrm{Ric}_\mu = \mathrm{M}_{\mu_{\mathfrak{p}}} - \tfrac{1}{2}\,\mathrm{B}_\mu,$$

where $\langle \mathrm{B}_\mu \cdot, \cdot \rangle := \mathrm{B}_\mu|_{\mathfrak{p}\times\mathfrak{p}}$ Killing form, and M is the moment map for the representation $\theta : \mathfrak{gl}(\mathfrak{p}) \to \mathrm{End}(\Lambda^2\mathfrak{p}^* \otimes \mathfrak{p})$,

$$\theta(A)\lambda := A\lambda(\cdot,\cdot) - \lambda(A\cdot,\cdot) - \lambda(\cdot, A\cdot),$$

$$\text{i.e.,} \quad \langle \mathrm{M}_{\mu_{\mathfrak{p}}}, A \rangle := \tfrac{1}{4}\langle \theta(A)\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}} \rangle, \quad \forall A \in \mathfrak{gl}(\mathfrak{p}) \quad (\mathit{GIT}).$$

The scalar curvature of g is $\mathrm{Sc}_\mu = -\frac{1}{4}|\mu_{\mathfrak{p}}|^2 - \frac{1}{2}\,\mathrm{tr}\,\mathrm{B}_\mu$.

Consider the maps

$$\delta_{\mu_{\mathfrak{p}}} : \mathfrak{gl}(\mathfrak{p}) \longrightarrow \Lambda^2\mathfrak{p}^* \otimes \mathfrak{p}, \qquad \delta^t_{\mu_{\mathfrak{p}}} : \Lambda^2\mathfrak{p}^* \otimes \mathfrak{p} \longrightarrow \mathfrak{gl}(\mathfrak{p}),$$

where $\delta_{\mu_{\mathfrak{p}}}(A) := -\theta(A)\mu_{\mathfrak{p}}$ and $\delta^t_{\mu_{\mathfrak{p}}}$ is the transpose of $\delta_{\mu_{\mathfrak{p}}}$, and

the operator $C_{\mu_{\mathfrak{p}}} : \mathrm{sym}(\mathfrak{p}) \longrightarrow \mathrm{sym}(\mathfrak{p})$,

$$C_{\mu_{\mathfrak{p}}}(A) := S \circ \delta^t_{\mu_{\mathfrak{p}}}\delta_{\mu_{\mathfrak{p}}}(A) + 2A\,\mathrm{M}_{\mu_{\mathfrak{p}}} + 2\,\mathrm{M}_{\mu_{\mathfrak{p}}}A.$$

- $C_{\mu_{\mathfrak{p}}}(I) = 0 \quad (\langle \mathrm{M}_{\mu_{\mathfrak{p}}}, A\rangle = \frac{1}{4}\langle\theta(A)\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}}\rangle = -\frac{1}{4}\langle A, \delta^t_{\mu_{\mathfrak{p}}}(\mu_{\mathfrak{p}})\rangle)$.
- $C_{\mu_{\mathfrak{p}}}(\mathrm{sym}(\mathfrak{p})) \perp \mathbb{R}I$.
- $C_{\mu_{\mathfrak{p}}}$ is a self-adjoint operator.
- $\langle C_{\mu_{\mathfrak{p}}}(A), A\rangle = |\theta(A)\mu_{\mathfrak{p}}|^2 + \langle\theta(A^2)\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}}\rangle$ for any $A \in \mathrm{sym}(\mathfrak{p})$.
- $C_{\mu_{\mathfrak{p}}}\left(\mathrm{sym}(\mathfrak{p})^K\right) \subset \mathrm{sym}(\mathfrak{p})^K$.
- Relation with Chevalley cohomology (K trivial, so $\mathfrak{p} = \mathfrak{g}$),

$$\mathfrak{g} \underset{\mathrm{ad}_\mu}{\longrightarrow} \mathfrak{gl}(\mathfrak{g}) \underset{\delta_\mu}{\longrightarrow} \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}, \;\; \Delta_\mu := \mathrm{ad}_\mu \mathrm{ad}^t_\mu + \delta^t_\mu\delta_\mu, \;\; \mathrm{Ker}\,\Delta_\mu = H^1(\mathfrak{g}, \mathfrak{g}).$$

First variation of Ricci and the Lichnerowicz Laplacian

Recall $C_{\mu_\mathfrak{p}}(A) := S \circ \delta_{\mu_\mathfrak{p}}^t \delta_{\mu_\mathfrak{p}}(A) + 2A\,\mathrm{M}_{\mu_\mathfrak{p}} + 2\,\mathrm{M}_{\mu_\mathfrak{p}}\,A$.

Lemma ([L.-Will, 20])

For any $T = \langle A\cdot, \cdot\rangle \in \mathrm{sym}^2(\mathfrak{p})^K$, $A \in \mathrm{sym}(\mathfrak{p})^K$,

$$d\,\mathrm{Rc}\,|_g\, T = \tfrac{1}{4}\langle C_{\mu_\mathfrak{p}}(A)\cdot, \cdot\rangle.$$

[Changliang Wang-McKenzie Wang 18] $\mathcal{S}^2(M)^G \subset \mathrm{Ker}\,\delta_g$ for G compact, which implies that $d\,\mathrm{Rc}\,|_g = \frac{1}{2}\Delta_L$ on the subspace $\mathcal{S}^2(M)^G$, where Δ_L is the Lichnerowicz Laplacian of g.

Corollary ([L.-Will, 20])

If G compact, then

$$\Delta_L T = 2d\,\mathrm{Rc}\,|_g\, T = \tfrac{1}{2}\langle C_{\mu_\mathfrak{p}}(A)\cdot, \cdot\rangle.$$

First variation of Ricci and the Lichnerowicz Laplacian

Recall $C_{\mu_{\mathfrak{p}}}(A) := S \circ \delta_{\mu_{\mathfrak{p}}}^t \delta_{\mu_{\mathfrak{p}}}(A) + 2A\,\mathrm{M}_{\mu_{\mathfrak{p}}} + 2\,\mathrm{M}_{\mu_{\mathfrak{p}}} A$.

Lemma ([L.-Will, 20])

For any $T = \langle A\cdot, \cdot\rangle \in \mathrm{sym}^2(\mathfrak{p})^K$, $A \in \mathrm{sym}(\mathfrak{p})^K$,

$$d\,\mathrm{Rc}\,|_g T = \tfrac{1}{4}\langle C_{\mu_{\mathfrak{p}}}(A)\cdot, \cdot\rangle.$$

- $d\,\mathrm{Rc}\,|_g(\mathcal{S}^2(M)^G) \perp_g \mathbb{R}g$.
- $d\,\mathrm{Rc}\,|_g$ is a self-adjoint operator.
- If g Einstein, say $\mathrm{Rc}(g) = \rho g$, then $d\,\mathrm{Rc}\,|_g = 2\rho\,\mathrm{id}$ restricted to $T_g\,\mathrm{Aut}(G/K)\cdot g$, recall

$$T_g\mathcal{M}^G = \mathcal{S}^2(M)^G = (\mathbb{R}g \oplus T_g\,\mathrm{Aut}(G/K)\cdot g) \oplus^{\perp_g} W_g.$$

Naturally reductive case

$g \in \mathcal{M}^G$ naturally reductive with respect to G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, i.e., $\operatorname{ad}_{\mathfrak{p}} X : \mathfrak{p} \to \mathfrak{p}$ skew-symmetric $\forall X \in \mathfrak{p}$, where $\operatorname{ad}_{\mathfrak{p}} X : \mathfrak{p} \to \mathfrak{p}$, $Y \mapsto \mu_{\mathfrak{p}}(X, Y)$ ($\Leftrightarrow \exp tX \cdot o$ is a geodesic). Then,

$$\mathrm{M}_{\mu_{\mathfrak{p}}} = \tfrac{1}{4} \sum (\operatorname{ad}_{\mathfrak{p}} X_i)^2,$$

where $\{X_i\}$ is any orthonormal basis of $(\mathfrak{p}, \langle \cdot, \cdot \rangle)$.

Remark. For G compact semisimple, K trivial ($\mathfrak{p} = \mathfrak{g}$) and $\langle \cdot, \cdot \rangle = -\mathrm{B}_{\mu}$, $\mathrm{M}_{\mu} = -\frac{1}{4}$ Casimir operator acting on the adjoint representation of $\mathfrak{g}$.

$$\mathrm{C}_{\mu_{\mathfrak{p}}}(A) := -\sum [\operatorname{ad}_{\mathfrak{p}} X_i, [\operatorname{ad}_{\mathfrak{p}} X_i, A]].$$

Note that $\mathrm{C}_{\mu_{\mathfrak{p}}} \geq 0$ and $\mathrm{C}_{\mu_{\mathfrak{p}}}(A) = 0$ if and only if $[A, \operatorname{ad}_{\mathfrak{p}} \mathfrak{p}] = 0$.

Remark. $\mathrm{C}_{\mu} =$ Casimir operator acting on the representation $\operatorname{sym}(\mathfrak{g})$ of $\mathfrak{g}$ given by $\tau(X)A := [\operatorname{ad} X, A]$, i.e., $\mathrm{C}_{\mu} = -\sum \tau(X_i)^2$.

Naturally reductive case

Theorem ([L.-Will, 20])

Let $g \in \mathcal{M}^G$ be a de Rham irreducible metric. If g is naturally reductive with respect to G and $\operatorname{Sc}(g) \neq 0$, then g is Ricci locally invertible.

Proof.

We use that if g is irreducible, then there exists no nontrivial subspace invariant under the space of operators $\operatorname{ad}\mathfrak{k}|_{\mathfrak{p}} + \operatorname{ad}_{\mathfrak{p}}\mathfrak{p}$ (Kostant 56, D'Atri-Ziller 79). Thus for $A \in \operatorname{sym}(\mathfrak{p})^K$,

$$\mathrm{C}_{\mu_{\mathfrak{p}}}(A) = -\sum [\operatorname{ad}_{\mathfrak{p}} X_i, [\operatorname{ad}_{\mathfrak{p}} X_i, A]] = 0 \;\Leftrightarrow\; [\operatorname{ad}_{\mathfrak{p}} X_i, A] = 0 \;\forall i \;\Leftrightarrow\; A = aI,$$

that is, $\operatorname{Ker}\mathrm{C}_{\mu_{\mathfrak{p}}} = \mathbb{R}I$. But therefore $\operatorname{Ker} d\operatorname{Rc}|_g = \mathbb{R}g$ by the above lemma and so g is Ricci locally invertible by our first theorem. □

Theorem ([L.-Will, 20])

Let $g \in \mathcal{M}^G$ be a de Rham irreducible metric. If g is naturally reductive with respect to G and $\mathrm{Sc}(g) \neq 0$, then g is Ricci locally invertible.

- For any compact G there is a normal metric $g \in \mathcal{M}^G$, which is of course naturally reductive and has $\mathrm{Sc}(g) > 0$. If in addition $M \neq G_1/K_1 \times G_2/K_2$, then g is irreducible and so Ricci locally invertible.
- [D'Atri-Ziller 79] If $M = H$ is a compact semisimple Lie group and $G = H \times K$, where $K \subset H$, then any $g \in \mathcal{M}^G$ (i.e., any left-invariant g on H which is in addition $\mathrm{Ad}(K)$-invariant) is naturally reductive w.r.t. G. For H simple g is always irreducible, so g is Ricci locally invertible as soon as $\mathrm{Sc}(g) \neq 0$.
- [D'Atri-Ziller 79, Gordon 85] Any $\mathrm{Ad}(K)$-invariant metric on a non-compact semisimple $M = H$, $K \subset H$ maximal compact, is also naturally reductive w.r.t. $G = H \times K$ If H simple and $\mathrm{Sc}(g) \neq 0$ then g is Ricci locally invertible.

- [Gordon 85] Two-step nilpotent Lie groups N attached to representations of compact Lie groups admit naturally reductive metrics w.r.t. $G = K \ltimes N$, where $K = \mathrm{Aut}(\mathfrak{n}) \cap \mathrm{O}(\mathfrak{n})$. If N indecomposable then Ricci locally invertible.
- [Gordon 85] Structure theorem for naturally reductive spaces: they are all "amalgamated" constructions, in a very specified way, of the above three types.
- [Kowalski, Tricerri, Vanhecke, 80s] [Agricola, Ferreira, Friedrich, Storm, 2015-] Classification results of naturally reductive spaces in dimension ≤ 8.
- If $M = G$ is a 2-step nilpotent Lie group with non-trivial abelian factor (i.e., $[\mathfrak{n}, \mathfrak{n}] \neq \mathfrak{z}(\mathfrak{n})$), then $\mathcal{M}^G_{inv}$ is empty.
- $\mathcal{M}^G_{inv}$ is open and dense for $M = G = \mathrm{SU}(2) \times S^1$.

Many thanks for your attention.

Happy Birthday Dmitri !!