Discrete Lipschitz-Killing curvatures

Ivan Izmestiev

TU Wien

Conference in honor of the 80th birthday of Dmitri Alekseevsky

Table of Contents

Classical Lipschitz-Killing curvatures

(M, g): a Riemannian manifold, R_g: the Riemann curvature tensor.

Lipschitz-Killing curvatures are the integrals

$$S_{2k}(g) := \int_M \mathrm{tr}(R_g^k)\, \mathrm{dvol}_g, \quad k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,$$

where $R_g^k \colon \Lambda^{2k} T_p M \to \Lambda^{2k} T_p M$.

In particular,

$$S_0(g) = \mathrm{vol}(g), \quad S_2(g) = \int_M \mathrm{scal}_g\, \mathrm{dvol}_g$$

For n even, S_n is proportional to the Euler characteristic (Chern-Gauss-Bonnet theorem), thus independent of g.

Weyl's tube formula

$M \subset \mathbb{R}^p$: an n-dimensional submanifold, g: the induced metric on M.

Theorem (Weyl)

For all r sufficiently small, the volume of the r-neighborhood (the tube) around M is a polynomial in r with coefficients proportional to the Lipschitz-Killing curvatures of (M, g):

$$\mathrm{vol}(B_r(M)) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c(p, n, k) S_{2k}(g) r^{p-n+2k}$$

In particular, the coefficients depend on g only.

Example: for a surface in $\mathbb{R}^3$ one has

$$\mathrm{vol}(B_r(M)) = 2r\,\mathrm{area}(M) + \frac{4\pi}{3} r^3 \chi(M).$$

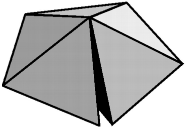
Euclidean cone-manifolds

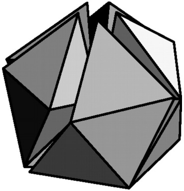
... also known as piecewise flat spaces or polyhedral manifods, are discrete analoga of Riemannian manifolds.

Constructive definition

A Euclidean cone-manifold is a manifold glued from Euclidean polyhedra by isometries between their faces.

Example 1: gluing regular tetrahedra.



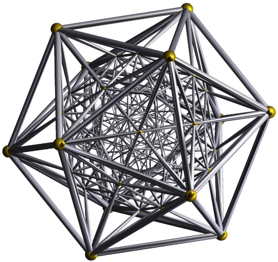


If one proceeds by surrounding each vertex by 20 tetrahedra, one gets...

Euclidean cone-manifolds: examples

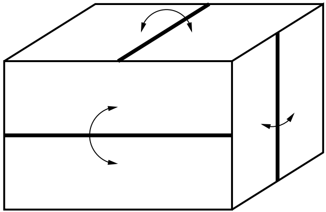
...a Euclidean cone-metric on $\mathbb{S}^3$, the boundary of the 600-cell, a regular 4-dimensional polyhedron.

The skeleton of the 600-cell:



Example 2: “Fold” each face of a parallelepiped as shown.

Get a Euclidean cone-metric on $\mathbb{S}^3$ with Borromean rings as the singular locus.

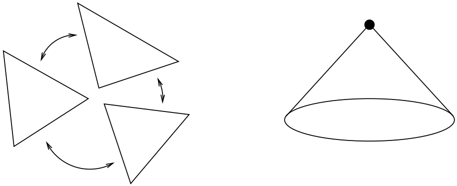


Euclidean cone-manifolds

Combinatorics is not important; important is the metric structure.

Descriptive definition

A Euclidean cone-manifold is a manifold with an atlas with values in certain model spaces (defined by induction on the dimension).



In dimension 2 a singular point is characterized by the angle $\omega \neq 2\pi$ around this point. The angle deficit $\kappa = 2\pi - \omega$ is called the curvature.

Discrete Gauss-Bonnet theorem

Theorem

For every Euclidean cone-metric on a closed surface M one has

$$\sum \kappa_i = 2\pi\chi(M).$$

Proof.

Triangulate the surface.

$$\left.\begin{array}{r} V - E + F = \chi(M) \\ 2E = 3F \end{array}\right\} \Rightarrow V - \frac{F}{2} = \chi(M)$$

$$\sum \omega_i = \sum(\alpha_j + \beta_j + \gamma_j) = \pi F$$

$$\sum \kappa_i = 2\pi V - \sum \omega_i = 2\pi\left(V - \frac{F}{2}\right) = 2\pi\chi(M)$$

□

Regge functional or discrete total scalar curvature

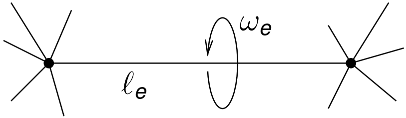
This is the discrete analog of the Einstein-Hilbert functional, i. e. of the total scalar curvature.

Definition

If $\dim M = 3$*, and* c *is a Euclidean cone-metric on* M*, then*

$$S_2^{\Delta}(c) = \sum_e \ell_e \kappa_e,$$

where the sum is over all “edges” of c*.*



Regge functional in higher dimensions

Definition

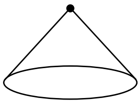
If $\dim M = n$*, and* c *is a Euclidean cone-metric on* M*, then*

$$S_2^{\Delta}(c) = \sum_{Q_i \subset \Sigma^{n-2}} \mathrm{vol}_{n-2}(Q_i)\kappa(M/Q_i).$$

The singular locus Σ of M is stratified:

$$\Sigma = \Sigma^0 \cup \Sigma^1 \cup \cdots \cup \Sigma^{n-2}.$$

Σ^k is the union of open k-manifolds Q_i; each Q_i has a “normal space” M/Q_i which is a cone-manifold; the neighborhood of every point $p \in Q_i$ is isometric to $U \times (M/Q_i), U \subset \mathbb{R}^k$.



For $k = n - 2$, M/Q_i is a 2-dimensional cone

Variational property of the Regge functional

Fix a triangulation of a cone-manifold M of $\dim M = 3$.
Vary the metric by changing the edge lengths.
Recall: $S_2(\ell) = \sum_e \ell_e \kappa_e$.

Theorem

$$\frac{\partial S_2^\Delta}{\partial \ell_e} = \kappa_e$$

Proof.

Follows from the Schläfli formula for a Euclidean tetrahedron:

$$\sum_e \ell_e d\omega_e = 0.$$

□

Variational property of the Regge functional

Formula $\frac{\partial S_2^\Delta}{\partial \ell_e} = \kappa_e$ implies that critical points of S_2 corresponds to Euclidean metrics on M.

Compare this to the first variation of the Einstein-Hilbert functional:

$$\frac{\partial}{\partial t}\bigg|_{t=0} S_2(g + th) = \int_M \left\langle \frac{\mathrm{scal}_g}{2} g - \mathrm{Ric}_g, h \right\rangle \mathrm{dvol}_g$$

In dimensions $n > 2$, critical points of S_2 are Ricci-flat metrics.
For $n = 3$ Ricci-flat means flat (Euclidean).

On the space of metrics of fixed volume, critical points are Einstein metrics $\mathrm{Ric} = \lambda g$. Again, for $n = 3$ this reduces to $\sec = \mathrm{const}$.

Geometrization of 3-manifolds

Theorem (Geometrization theorem)

Closed 3-dimensional manifolds can be cut into pieces carrying one of the eight standard geometries.

An approach based on variational properties of the Einstein-Hilbert functional was being developed in 1990's.
This was preceded by Yamabe's attempt to solve the Poincaré conjecture with a similar approach.

The Regge functional approach: tempting but meets with difficulties.
A dual approach: Casson–Rivin, variational properties of the volume of ideal hyperbolic tetrahedra.

Geometrization with boundary conditions

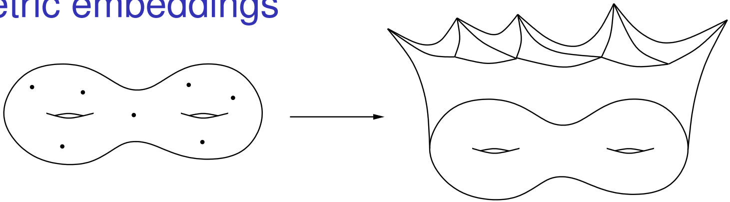
Open 3-manifolds (irreducible, atoroidal) may carry infinitely many different hyperbolic structures.
One can try to fix the structure by fixing its behavior at infinity or on the boundary.

Theorem (Fillastre'07)

Given a hyperbolic cone-metric with positive singular curvatures on a surface of genus > 1*, there is a unique Fuchsian manifold with convex polyhedral boundary carrying this metric.*

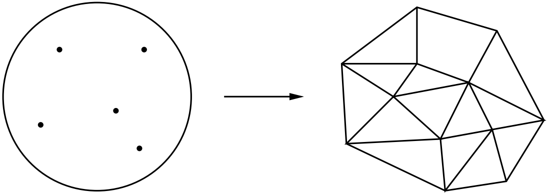
[Prosanov'20]: a variational proof based on the Regge functional.
Luo, Springborn,...: this and similar theorems can be interpreted as "discrete uniformization".

Isometric embeddings



Theorem (Alexandrov)

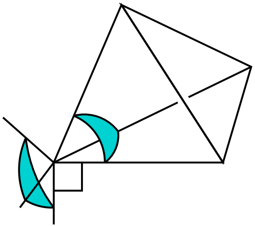
Given a Euclidean cone-metric with positive singular curvatures on the sphere, there is a unique convex Euclidean polyhedron whose boundary carries this metric.



External angles

For a simplex σ and a vertex v of σ define

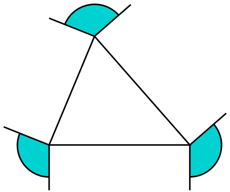
- the (normalized) internal angle $\alpha(v, \sigma)$;
- the (normalized) external angle $\beta(v, \sigma)$.



Theorem (Discrete Gauss-Bonnet-Hopf theorem)

For every simplex σ one has

$$\sum_{v \in \sigma} \beta(v, \sigma) = 1.$$



Discrete Chern-Gauss-Bonnet theorem

Definition

Let (M, c) be a Euclidean cone-manifold and T its triangulation. Define the Chern-Gauss-Bonnet density as a function on the vertex set of T

$$r(v) = \sum_{\sigma \ni v} (-1)^{\dim \sigma} \beta(v, \sigma).$$

Theorem

$$\sum_v r(v) = \chi(M)$$

Proof.

$$\sum_v r(v) = \sum_v \sum_\sigma (-1)^{\dim \sigma} \beta(v, \sigma) = \sum_\sigma (-1)^{\dim \sigma} \sum_v \beta(v, \sigma)$$

$$= \sum_\sigma (-1)^\sigma = \sum_{k=0}^{n} (-1)^k f_k(T) = \chi(T) = \chi(M)$$

Properties of the CGB density

[Cheeger, Müller, Schrader'86]: On the curvature of piecewise flat spaces

1. Function r is independent of the choice of a triangulation T.
2. If a sequence c_n of cone-metrics converges to a Riemannian metric g in a good way (there are triangulations T_n of c_n whose simplices are not too thin), then r_n converges to the Riemannian Chern-Gauss-Bonnet density.
3. If (M, c) is embedded as a polyhedral hypersurface, then $r(v)$ is equal to the external angle at v.

Other Lipschitz-Killing densities are defined as

$$r(\tau) = \sum_{\sigma \supset \tau} (-1)^{\dim \sigma - \dim \tau} \beta(\tau, \sigma)$$

and have similar properties.

Peter McMullen's identities

Let v be a vertex of a simplex σ. Then one has

$$\sum_{v \le \tau \le \sigma} \alpha(v,\tau)\beta(\tau,\sigma) = 1$$

$$\sum_{v \le \tau \le \sigma} (-1)^{\dim \tau} \alpha(v,\tau)\beta(\tau,\sigma) = 0$$

Can use any of these to express the external angle $\beta(v,\sigma)$ in terms of internal angles and external angles of smaller dimension. For example,

$$\beta(v,\sigma) = -\sum_{v \le \tau < \sigma} (-1)^{\dim \tau} \alpha(v,\tau)\beta(\tau,\sigma)$$

Another formula for the discrete CGB density

Substituting $\beta(v,\sigma) = -\sum_{v \le \tau < \sigma} (-1)^{\dim \tau} \alpha(v,\tau)\beta(\tau,\sigma)$ into $r(v) = \sum_{\sigma \ni v} (-1)^{\dim \sigma} \beta(v,\sigma)$ one gets

$$r(v) = 1 - \sum_{\sigma > v} \alpha(v,\sigma) r(\sigma).$$

By induction,

$$r(v) = \sum_{k \ge 0} \sum_{v < \sigma_1 < \cdots < \sigma_k} (-1)^k \alpha(v,\sigma_1)\alpha(\sigma_1,\sigma_2)\cdots\alpha(\sigma_{k-1},\sigma_k).$$

Intrinsically,

$$r(v) = \sum_{k \ge 0} \sum_{v < Q_1 < \cdots < Q_k} (-1)^k \alpha(v,Q_1)\alpha(Q_1,Q_2)\cdots\alpha(Q_{k-1},Q_k).$$

Yet another formula for the discrete CGB density

Adding the McMullen identities

$$\sum_{v \le \tau \le \sigma} \alpha(v,\tau)\beta(\tau,\sigma) = 1$$

$$\sum_{v \le \tau \le \sigma} (-1)^{\dim \tau} \alpha(v,\tau)\beta(\tau,\sigma) = 0$$

one gets rid of τ with $\dim \tau$ odd and obtains the formula

$$r(v) = \sum_{k \ge 0} \sum_{v < Q_1 < \cdots < Q_k} (-1)^k \alpha(v, Q_1)\alpha(Q_1, Q_2) \cdots \alpha(Q_{k-1}, Q_k),$$

where the summation goes only over the even-dimensional strata.

This generalizes the formula for $n = 2$: $r(v) = \kappa = 1 - \alpha(v, Q)$, where Q is the (unique) 2-dim stratum adjacent to v.

References

- T. Regge: General relativity without coordinates
- J. Cheeger, W. Müller, R. Schrader: On the curvature of piecewise flat spaces
- J. Lafontaine: Mesures de courbures des variétés lisses et des polyèdres
- P. McMullen: Non-linear angle-sum relations for polyhedral cones and polytopes
- L. Budach: Lipschitz-Killing curvatures of angular partially ordered sets
- I. Izmestiev: Variational properties of the discrete Hilbert-Einstein functional