Clifford algebras and engineering applications

GEOMETRY AND APPLICATIONS ONLINE

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Grassmann algebra $(\mathbb{G}r_n, \wedge)$

$$\mathbb{R}^n,\ \ e_1, \dots, e_n \text{ is set of basis elements}$$

Free associative anticommutative distributive algebra over $e_1, \dots, e_n$ is called Grassmann algebra $\mathbb{G}r_n$ together with bilinear product $\wedge$.

Linear subspace $A \in \mathbb{G}r_n$

$$x \in A \Leftrightarrow x \wedge A = 0$$

Example: $A = e_1 \wedge e_2$

$$x \wedge A = \left(\sum x_i e_i\right) \wedge e_1 \wedge e_2 = \sum_{i \neq 1,2} x_i (e_i \wedge e_1 \wedge e_2)$$

$$x \wedge A = 0 \Leftrightarrow x_i = 0,\ \ i = 3, \dots, n \Leftrightarrow x = x_1 e_1 + x_2 e_2$$

$$\begin{aligned}\mathbb{G}r_n &= \mathbb{R} + \mathbb{R}^n + \wedge^2 \mathbb{R}^n + \cdots + \wedge^{n-1} \mathbb{R}^n + \wedge^n \mathbb{R}^n \\ &= \mathbb{R} + \text{ lines } + \text{ planes } + \cdots + \text{ hyperplanes } + \text{volume element}\end{aligned}$$

Clifford algebra $\mathbb{G}_n$

Euclidean scalar product $\cdot$ on $\mathbb{R}^n$, quadratic vector space $\mathbb{R}^n := \mathbb{R}^{n,0,0}$ defines to Clifford algebra $\mathbb{G}_n$ with signature $(n, 0, 0)$

Geometric product on vectors $\mathbb{R}^n \subset \mathbb{G}_n$

$u \cdot v = \frac{1}{2}(uv + vu),\ u \wedge v = \frac{1}{2}(uv - vu),\ uv = u \cdot v + u \wedge v$

Operations

$$
\begin{aligned}
u \wedge v &= \langle uv \rangle_{k+l}, \\
u \cdot v &= \langle uv \rangle_{|k-l|}, \\
u \lfloor v &= \langle uv \rangle_{k-l}, \\
u \rfloor v &= \langle uv \rangle_{l-k},
\end{aligned}
$$

$u \in \wedge^k \mathbb{R}^n,\ v \in \wedge^l \mathbb{R}^n$

Lie group of Versors

Reflection with respect to hyperplane perpendicular to $a \in \mathbb{R}^n$

$$x \mapsto x - \frac{2(x \cdot a)a}{||a||^2} = x - \frac{(xa + ax)a}{a^2} = axa^{-1},$$

$$G = \{a_1 \cdots a_l \mid a_i^2 = 1, a_i \in \mathbb{R}^n\} \text{ versors}$$

Example: $u, v \in \mathbb{R}^n$, $uv = u \cdot v + u \wedge v = \cos(u, v) + sin(u, v) u \wedge v$ rotation with respect to the plane $u \wedge v$.

Duality

Hodge duality

$$A \wedge A^* = (A \cdot A^*) e_1 \cdots e_n$$

algebraically $A^* = -Ae_1 \cdots e_n$
For example line $t \in \mathbb{R}^n$ is a dual to $(n-1)-$vector $-te_1 \cdots e_n$ which is hyperplane.

$$(x \wedge A)^* = x \cdot A^* \rightsquigarrow \text{ dual representation } x \in A^* \Leftrightarrow x \cdot A^* = 0$$

$A, B \in \mathbb{G}_n$

A is a linear subspace generated by $u_1, \ldots, u_{l_1}$ and B is a linear subspace generated by $v_1, \ldots, v_{l_2}$. Then

$$x \cdot (A^* \wedge B^*) = (x \cdot A^*) \wedge B^* + A^* \wedge (x \cdot B^*)$$

$$x \in (A^* \wedge B^*) \Leftrightarrow x \in A^* \text{ and } x \in B^*$$

So $\wedge$ is an intersection on dual representation.

Lie algebra T_eG

Curve $a_1(t)\cdots a_l(t)\in G$, such that $a_1(0)\cdots a_l(0)=e$:

$$\begin{aligned}
\partial_t(a_1(t)\cdots a_l(t)) &= \dot a_1(t)a_2(t)\cdots a_l(t)+a_1(t)\dot a_2(t)\cdots a_l(t)+\cdots\\
&+a_1(t)a_2(t)\cdots \dot a_l(t)\\
&=\dot a_1(t)a_1(t)a_1(t)a_2(t)\cdots a_l(t)+a_1(t)\dot a_2(t)a_2(t)a_2(t)\cdots a_l(t)+\cdots\\
&+a_1(t)a_2(t)\cdots \dot a_l(t)a_l(t)a_l(t)\\
&\Rightarrow^{t=0}\ \dot a_1(0)a_1(0)+\dot a_2(0)a_2(0)+\cdots+\dot a_l(t)a_l(t)\\
&=\dot a_1(0)\wedge a_1(0)+\dot a_2(0)\wedge a_2(0)+\cdots+\dot a_l(t)\wedge a_l(t)
\end{aligned}$$

because $a_i(t)^2=1\Rightarrow a_i(t)\cdot \dot a_i(t)=0$

$\rightsquigarrow\ T_eG\cong\wedge^2\mathbb{R}^n=\mathfrak{so}(n)$
Example: $\mathbb{G}_3$, $\wedge^2\mathbb{R}^3=\mathrm{Im}\mathbb{H}$, Versor group $G=Spin(3)$

Rigid body motion

Lie group and Lie algebra

Lie group

$$Spin(n) \ltimes \mathbb{R}^n \to^{2:1} \to SO(n) \ltimes \mathbb{R}^n,$$

Lie algebra $\mathfrak{so}(n) \ltimes \mathbb{R}^n$, dimension $\frac{(n)(n-1)}{2} + n = \frac{(n+1)(n)}{2} = \binom{n+1}{2}$.

$\rightsquigarrow \wedge^2 \mathbb{R}^{n+1} \cong \mathfrak{so}(n) \ltimes \mathbb{R}^n$, basis $e_1, \ldots, e_n, e$, such that
$\wedge^2 \langle e_1, \ldots, e_n \rangle \cong \mathfrak{so}(n)$
and
$e \wedge \langle e_1, \ldots, e_n \rangle \cong \mathbb{R}^n$ is commutative subalgebra, so
$0 = [e \wedge e_1, e \wedge e_2] = ee_1 ee_2 - ee_2 ee_1 = e^2(2e_2 e_1) \Rightarrow e^2 = 0$

Affine extension (PGA)

$$\iota : \mathbb{R}^n \hookrightarrow \mathbb{G}_{n,0,1},$$

$A \in \wedge^2 \mathbb{R}^{n+1}, \exp(tA) : \iota(\mathbb{R}^n) \to \iota(\mathbb{R}^n) \in Spin(n) \ltimes \mathbb{R}^n,$
$A = e \wedge t, t \in \langle e_1, \dots, e_n \rangle$ {translation}

$$exp(tA)\iota(0)\exp(-At) = \iota(0 + t)$$

The first hint can be $\iota(0) = e$, but
$\exp(tA) e \exp(-At) = (1 + \frac{1}{2} e \wedge t) e (1 - \frac{1}{2} e \wedge t) = e$
The right choice is $\iota(0) = e_1 \cdots e_n$:
$\exp(tA) e_1 \cdots e_n \exp(-At) = (1 + \frac{1}{2} e \wedge t) e_1 \cdots e_n (1 - \frac{1}{2} e \wedge t) =$
$e_1 \cdots e_n + t_1 e e_2 \cdots e_n + t_2 e e_1 e_2 \cdots e_n + \cdots + e e_1 \cdots e_{n-1}$

$$\mathbb{R}^n \to \text{ hyperplanes}$$

One of the problems is a lack of duality $A^{**} = 0$

Conformal geometric algebra (CGA)

$$\mathbb{R}^n \hookrightarrow \mathbb{G}_{n+1,1,0}$$

The elements $e_1, \ldots, e_n$, e_+ and e_- such that $e_+^2 = 1$ and $e_-^2 = -1$
Introduce $e_0 = e_- + e_+$ and $e_\infty = \frac{1}{2}(e_- - e_+)$, such that $e_0^2 = e_\infty^2 = 0$ and $e_0 e_\infty + e_\infty e_0 = -2$. Two copies of affine extension PGA:
$CGA_0 = e_1, \ldots, e_n, e_0$ and $CGA_\infty = e_1, \ldots, e_n, e_\infty$

$$\mathbb{R}^n \hookrightarrow CGA_0, \quad \wedge^2 CGA_\infty \cong \mathfrak{so}(n) \ltimes \mathbb{R}^n$$

$$\begin{aligned} TeT^* &= \exp(e_\infty \wedge t) e_0 \exp(e_\infty \wedge t) = (1 + \frac{1}{2} e_\infty t) e_0 (1 - \frac{1}{2} e_\infty t) \\ &= e_0 - e_0 \frac{1}{2} e_\infty t + \frac{1}{2} e_\infty t e_0 - \frac{1}{2} e_\infty t e_0 \frac{1}{2} e_\infty t \\ &= e_0 - \frac{1}{2} t(e_0 e_\infty + e_\infty e_0) - \frac{1}{4} t^2 (-2 + e_0 e_\infty) e_\infty \\ &= e_0 + \frac{1}{2} t + \frac{1}{2} t^2 e_\infty \end{aligned}$$

$$t \hookrightarrow e_0 + t + \frac{1}{2} t^2 e_\infty =: t_c$$

CGA basic objects

$$t_c^2 = (e_0 + t + \frac{1}{2}t^2 e_\infty)^2 = -\frac{1}{2}t^2 + t^2 - \frac{1}{2}t^2 = 0 \text{ null cone}$$

$$t_1 \cdot t_2 = (e_0 + t_1 + \frac{1}{2}t_1^2 e_\infty) \cdot (e_0 + t_1 + \frac{1}{2}t_1^2 e_\infty)$$

$$= -\frac{1}{2}t_2^2 + t_1^2 - \frac{1}{2}t_1^2 = -\frac{1}{2}||t_2 - t_1||^2 \text{ norm linearisation}$$

$$e_\infty \cdot t = e_\infty \cdot (e_0 + t_1 + \frac{1}{2}t_1^2 e_\infty) = -1 \text{ normalisation}$$

Hyperplane as a bisector of two points P_1 and P_2

$$x \cdot P_1 = x \cdot P_2 \Rightarrow x \cdot (P_1 - P_2) = 0 \Rightarrow (P_1 - P_2)^* \text{hyperplane}$$

Sphere with the center c and radius ρ

$$x \cdot c = -\frac{1}{2}\rho^2 \Rightarrow x \cdot c = \frac{1}{2}\rho^2 (x \cdot e_\infty) \Rightarrow x \cdot (c - \frac{1}{2}\rho^2 e_\infty) = 0 \Rightarrow (c - \frac{1}{2}\rho^2 e_\infty)^* \text{ sphere}$$

Direct representation

A point pair (0D sphere), is defined by two points

$$P_1 \wedge P_2.$$

A circle (1D sphere) is defined by three points

$$P_1 \wedge P_2 \wedge P_3$$

or a point pair and a point. Finally, a sphere (2D sphere) is defined by four points

$$P_1 \wedge P_2 \wedge P_3 \wedge P_4$$

or two point pairs, etc. A plane and line can also be defined by points that lie on it and by the point at infinity, i.e. a line is represented by

$$P_1 \wedge P_2 \wedge e_\infty$$

and a plane by

$$P_1 \wedge P_2 \wedge P_3 \wedge e_\infty.$$

Dual representation

In the dual representation, a sphere can be represented by its center c and its radius ρ as

$$c - \frac{1}{2}\rho^2 e_\infty.$$

A plane is defined as

$$n + de_\infty,$$

where n is the unit normal vector of the plane and d is the distance to the origin.

Meet

In this sense, the wedge product is a constructive operator, i.e. $A \wedge B$ is an object spanned by A and B. The duality operator allows to define of the dual to wedge product, so called *meet*,

$$A \vee B = (A^* \wedge B^*)^*.$$

Geometrically, this gives a CGA representative of the intersection of objects A and B.

Rigid body motions in 3D

The translation in the direction $t = t_1 e_1 + t_2 e_2 + t_3 e_3$ is realized by the multivector (*translator*)

$$T = 1 - \frac{1}{2} t e_\infty$$

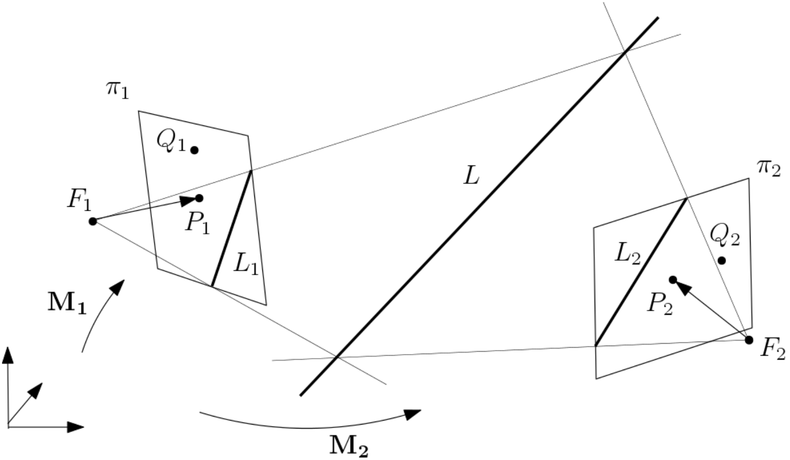
and the rotation around the origin and the normalized axis $L = L_1 e_1 + L_2 e_2 + L_3 e_3$ by an angle ϕ is realized by the multivector (*rotor*)

$$R = \mathrm{e}^{-\frac{1}{2} l \phi} = \cos \frac{\phi}{2} - l \sin \frac{\phi}{2},$$

where $l = L^*_{3D} = L(e_1 \wedge e_2 \wedge e_3) = L_1(e_2 \wedge e_3) + L_2(e_3 \wedge e_1) + L_3(e_1 \wedge e_2)$. The rotation around a general point and axis is then a composition $TR\tilde{T}$ of the translation to the origin, rotation R and reverse translation. A general composition of a translator with a rotor is called a motor.

Realisation

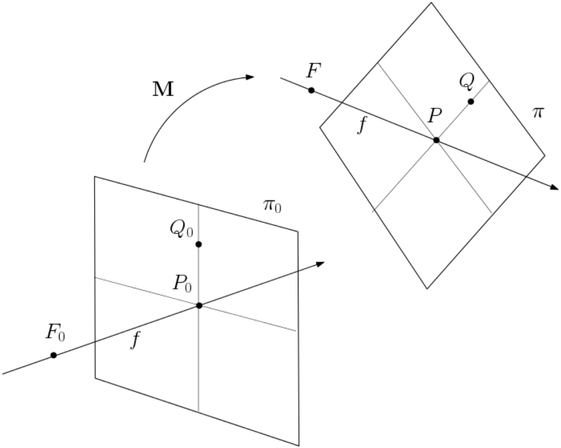
Pose estimation



$$L = (F_1 \wedge L_1) \vee (F_2 \wedge L_2).$$

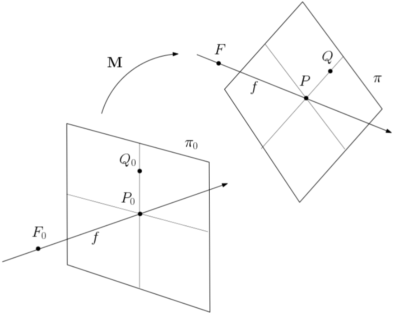
$$L_k = (L \wedge F_k) \vee \pi_k, \quad k = 1, 2,$$

Camera position



- focal distance $f = -2\sqrt{F \cdot P}$,
- camera direction $(F - P) \wedge e_\infty$,
- camera plane $\pi = P \wedge Q \wedge (F \wedge P \wedge e_\infty)^*$.

Camera position



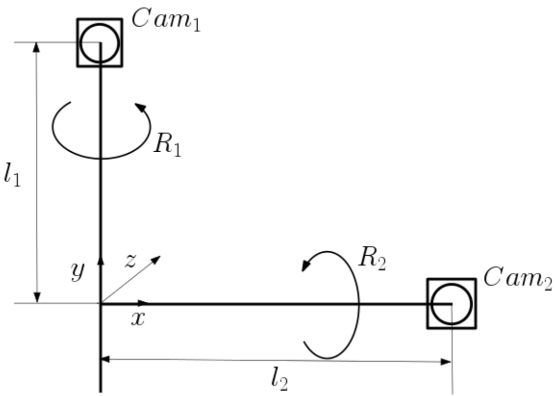
the actual position of the camera center is

$$F = MF_0\tilde{M}, \tag{1}$$

and the actual position of the image plane is given by

$$\pi = M\pi_0\tilde{M}. \tag{2}$$

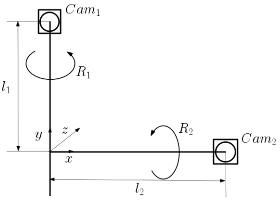
Realisation



In this case, the system can be described by the following set of motors.

$$M_1 = R_1 T_1,$$
$$M_2 = R_2 R_1 T_2,$$

Realisation



where the translations T_1, T_2 and the rotations R_1, R_2 are given by

$$T_1 = 1 - \frac{1}{2} l_1 e_2 \wedge e_\infty,$$

$$T_2 = 1 - \frac{1}{2} l_2 e_1 \wedge e_\infty,$$

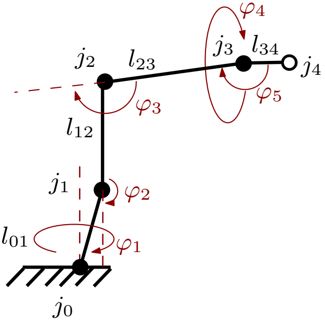
$$R_1 = \cos(\frac{\phi_1}{2}) + \sin(\frac{\phi_1}{2})(e_3 \wedge e_1),$$

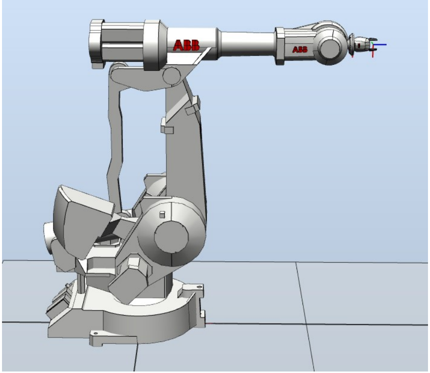
$$R_2 = \cos(\frac{\phi_2}{2}) + \sin(\frac{\phi_2}{2})\ell_2$$

and where the axis ℓ_2 of the second rotation is

$$\ell_2 = R_1(e_2 \wedge e_3)\tilde{R}_1.$$

ABB manipulator





$$J_i, \quad i = 0, \cdots, 4 \text{ joints}$$

$$P = j_4 + v \text{ orientation}$$

$$L_3^* = J_4 \wedge P \wedge e_\infty.$$

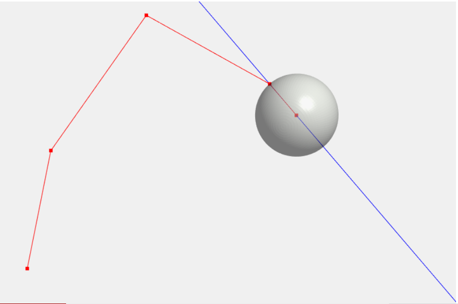
At first we compute J_3 with help of the intersection of the line L_3 and a sphere with center J_4 and radius l_{34}

$$S_3 = \bar{J}_4 - \frac{1}{2} l_{34}^2 e_\infty.$$

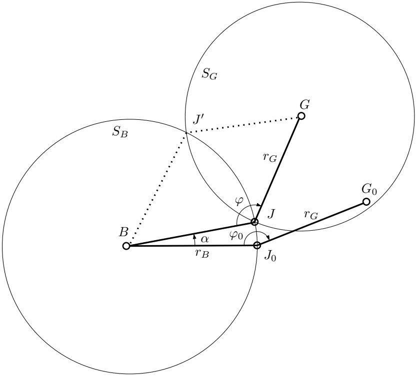
The intersection denotes the point pair Pp_3 and the corresponding point J_3 with respect to the orientation of the gripper is extracted:

$$Pp_3 = S_3 \wedge L_3,$$

$$\bar{J}_3 = \frac{-\sqrt{Pp_3^* \cdot Pp_3^*} + Pp_3^*}{-e_\infty \cdot Pp_3^*}.$$



Two link arm

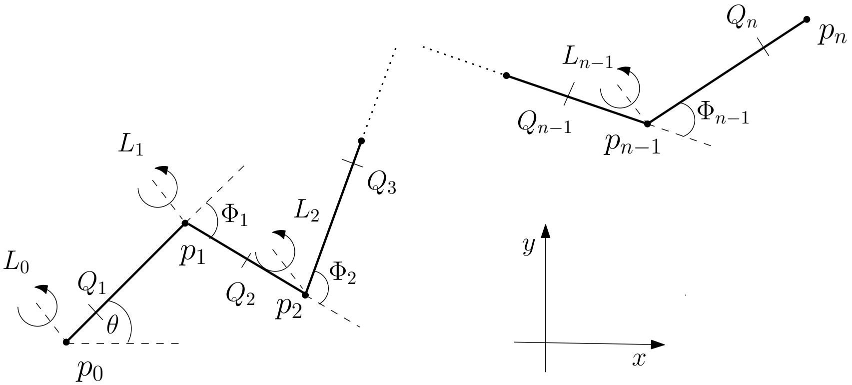


$$S_B = J_0 \cdot (B \wedge e_\infty), S_G = G - \tfrac{1}{2} r_G^2 e_\infty.$$

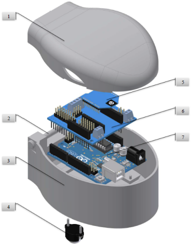
$$r_G = \sqrt{(J_0 \cdot (G_0 \wedge e_\infty)) \cdot (J_0 \cdot (G_0 \wedge e_\infty))}.$$

$$J' \wedge J = (S_B \wedge S_G)^*, J', J = (J' \wedge J \pm \sqrt{(J' \wedge J) \cdot (J' \wedge J)})(e_\infty \cdot (J' \wedge J)).$$

Robotic snake



$$p_i(q) = M_i p_i(0)\tilde{M}_i, M_0 = T_0 \mathrm{e}^{-\theta(e_1 \wedge e_2)}\tilde{T}_0, T_0 := 1 - \frac{1}{2}(xe_1 + ye_2)e_\infty,$$
$$\mathbf{M}_i = M_i \dots M_1 M_0 T_0 \text{ for } i > 0,$$
$$M_{i+1} = T_i \mathrm{e}^{-\Phi_i(e_1 \wedge e_2)}\tilde{T}_i, T_i = \mathrm{e}^{-(L_i - e_0)\wedge e_\infty}, L_i = \mathbf{M}_i L_i(0)\tilde{\mathbf{M}}_i,$$
$$Q_i = \mathbf{M}_i Q_i(0)\tilde{\mathbf{M}}_i. \quad (3)$$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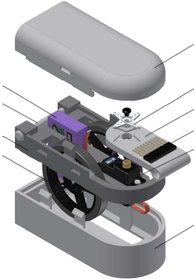

Thank you for your attention
Happy Birthday Dmitri
Všechno nejlepší Dmitri