An overview of G_2-structures and *Spin*(7)-structures on homogeneous spaces

Anna Fino

Dipartimento di Matematica
Universitá di Torino

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G_2-structures

Definition

A G_2-structure on a 7-manifold M^7 is given by a 3-form φ with pointwise stabilizer isomorphic to G_2.

- Pointwise $\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$.
- φ is non-degenerate: $i_X\varphi \wedge i_X\varphi \wedge \varphi \neq 0$, for every $X \neq 0$.

$\rightsquigarrow$ φ induces a metric g_φ with a volume form dV_φ:

$$g_\varphi(X, Y)dV_\varphi = \frac{1}{6} i_X\varphi \wedge i_Y\varphi \wedge \varphi.$$

Proposition (Fernández-Gray)

The following are equivalent:

(a) $\nabla^{LC}\varphi = 0$;

(b) $d\varphi = 0$ and $d(*\varphi) = 0$;

(c) $Hol(g_\varphi)$ is isomorphic to a subgroup of G_2.

A G_2-structure satisfying $(a), (b)$ or (c) is called parallel.

Remark

- The conditions $\nabla^{LC}\varphi = 0$ and $d(*\varphi) = 0$ are non-linear in φ.
- Metrics induced by parallel G_2-structures are Ricci-flat [Bonan].

The intrinsic torsion of a G_2-structure

The intrinsic torsion of a G_2-structure φ can be identified with

$$\nabla^{LC}\varphi \in T^* \otimes \mathfrak{g}_2^{\perp} \cong \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$$
$$\cong \mathbb{R} \oplus \mathfrak{g}_2 \oplus S_0^2(\mathbb{R}^7) \oplus \mathbb{R}^7$$

Some Fernández-Gray classes:

class	type	conditions
$\mathcal{X}_1$	nearly parallel	$d\varphi = k * \varphi$
$\mathcal{X}_1 \oplus \mathcal{X}_3$	coclosed or cocalibrated	$d * \varphi = 0$
$\mathcal{X}_2$	closed or calibrated	$d\varphi = 0$

Ricci tensor and Einstein condition

• The intrinsic torsion is entirely encoded into the exterior derivatives $d\varphi$, $d*\varphi$ as

$$\begin{aligned} d\varphi &= \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3, \\ d*\varphi &= 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi \end{aligned}$$

where τ_i is an intrinsic torsion i-form.

• The Ricci tensor $\mathrm{Ric}(g_\varphi)$ of (M^7, φ) can be expressed in terms of the intrinsic torsion forms τ_i [Bryant].

Problem

For which G_2-structures φ the metric g_φ can be Einstein?

- If φ is nearly-parallel ($d\varphi = \tau_0 * \varphi$), then g_φ is Einstein with $Scal(g_\varphi) = \frac{21}{8}\tau_0^2 > 0$.
- $\exists$ coclosed (not nearly parallel) G_2-structures φ with g_φ Einstein: the canonical G_2-structure of a 3-Sasakian manifold is coclosed [Agricola, Friedrich].
- If φ is closed, then $Scal(g_\varphi) = -\frac{1}{2}|\tau_2|^2 \leq 0 \rightsquigarrow$ no restrictions on compact manifolds!

Theorem (Cleyton-Ivanov; Bryant)

If M is compact with a closed G_2-structure φ with g_φ Einstein $\Rightarrow$ φ is parallel.

The characteristic connection

Theorem (Friedrich-Ivanov)

(M^7, φ) has a G_2-connection ∇^c with totally skew symmetric torsion $\iff \tau_2 = 0$.

- The resulting torsion 3-form is $T = \frac{7}{6}\tau_0\varphi - *d\varphi + *(4\tau_1 \wedge \varphi)$. and ∇^c admits (at least) one parallel spinor.

- M^7 with coclosed G_2-structures, $\nabla^c T = 0$ and non-abelian holonomy $\mathfrak{hol}(\nabla^c) \neq \mathfrak{g}_2$ have been classified [Friedrich].

$\Rightarrow$ By the classification $\exists$ G_2-structures with parallel characteristic torsion not inducing naturally reductive metrics.

G_2-structures on homogeneous spaces

Given (M^7, φ) consider the automorphism group

$$Aut(M^7, \varphi) := \{f \in Diff(M^7) | f^*\varphi = \varphi\}$$

If M^7 is compact, then $\mathfrak{aut}(M^7, \varphi) = \{X \in \chi(M^7) | L_X\varphi = 0\}$.

Remark

If a compact M^7 has a parallel G_2-structure φ with $Hol(g_\varphi) = G_2$, then $\mathfrak{aut}(M^7, \varphi) = \{0\}$.

Problem

If φ is not parallel, what can we say about $Aut(M^7, \varphi)$?

Nearly-parallel G_2-structures

Theorem (Friedrich, Kath, Moroianu, Semmelmann)

If a compact M^7 has a nearly parallel G_2-structure with $\dim Aut(M^7, \varphi) \geq 10$, then M^7 is homogeneous.

Simply connected compact homogeneous spaces with an invariant nearly parallel G_2-structure have been classified [Friedrich, Kath, Moroianu, Semmelmann].

Remark

Given (M^7, φ) with φ nearly parallel, the metric cone $(M^7 \times \mathbb{R}^+, \tilde{g} = r^2 g + dr^2)$ has holonomy contained in $Spin(7)$.

Example (Aloff-Wallach spaces)

Let $U(1)_{k,l}$ be the subgroup of $SU(3)$ generated by

$$\mathrm{diag}(e^{ik}, e^{il}, e^{i(-k-l)}).$$

• The spaces $N(k,l) = SU(3)/U(1)_{k,l}$, $(k,l) \neq (1,1)$, have 2 invariant nearly parallel G_2-structures such that $Hol(\tilde{g}) = Spin(7)$.

• For $k \neq l$, $N_{k,l}$ has a 2-dimensional family of invariant coclosed G_2-structures such that $d\varphi \wedge \varphi = 0$ [Cabrera, Monar, Swann].

• They admit a 4-dimensional family of invariant coclosed G_2-structures, including the 2 nearly parallel G_2-structures [Reidelgeld].

Coclosed G_2-structures

• Every compact spin M^7 admits a coclosed G_2-structure [Cronwley, Nordstrom].

• Every homogeneous coclosed G_2-structure can be extended to a cohomogeneity one parallel $Spin(7)$-structure and conversely every principal orbit of a parallel $Spin(7)$-manifold of cohomogeneity one has a homogeneous coclosed G_2-structure [Hitchin].

• Compact homogeneous manifolds with invariant coclosed G_2-structures have been classified [Reidelgeld].

• For Lie algebras there are some classification results [Freibert; Bagaglini, Fernández, F; Del Barco, Moroianu, Raffero].

Closed G_2-structures

Theorem (Podestá-Raffero)

M compact with φ closed non-parallel. If $X \in \mathfrak{aut}(M, \varphi)$, then the 2-form $i_X \varphi$ is harmonic. Consequently:
- $\dim \mathfrak{aut}(M, \varphi) \leq b_2(M)$;
- $\mathfrak{aut}(M, \varphi)$ *is abelian with dim* ≤ 6.

Consequences:

- There are no compact homogeneous examples with invariant (non-parallel) closed G_2-structures.
- Cohomogeneity one examples only occur for $M = \mathbb{T}^7$.

Non-compact homogeneous examples are provided by simply connected Lie groups G with a left-invariant closed G_2-structure [Fernández, Bryant, Lauret, Nicolini; F, Raffero; Ball, ...]

Remark

• Nilpotent Lie algebras with a closed G_2-structure have been classified [Conti, Fernández].

• No general classification is known for solvable Lie algebras admitting a closed G_2-structure.

• If the Lie groups G admit a lattice Γ they determine compact locally homogeneous spaces $(\Gamma\backslash G, \varphi)$, with φ invariant.

A classification result for the non-solvable case

Theorem (F-Raffero)

If $\mathfrak{g}$ is a non-solvable unimodular Lie algebra admitting closed G_2-structures and $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ its Levi decomposition, then $\mathfrak{s} \cong \mathfrak{sl}(2, \mathbb{R})$ and

- *if $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r} \Rightarrow \mathfrak{r}$ is centerless (3 Lie algebras up to isomorphism)*
- *if $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r} \Rightarrow \mathfrak{r} \cong \mathbb{R} \ltimes \mathbb{R}^3$ (1 Lie algebra up to isomorphism)*

Remark

A unimodular Lie algebra with a symplectic structure is solvable [Chu; Lichnerowicz-Medina].

Einstein condition on non-compact homogeneous case

Problem

Do there exist non-compact examples with a closed G_2 structure φ such that g_φ Einstein?

All the known examples of non-compact homogeneous Einstein manifolds are solvmanifolds. i.e. simply connected solvable Lie groups S endowed with a left-invariant metric.

Conjecture (Alekseevsky)

The Einstein solvmanifolds might exhaust the class of non-compact homogeneous Einstein manifolds!

In dimension seven the conjecture is true [Arroyo-Lafuente].

Theorem (Fernández-F-Manero)

Let g_φ determined by a left-invariant closed (or coclosed) G_2-structure φ on a solvmanifold. Then g_φ is Einstein if and only if g_φ is flat.

Spin(7)-structures

Spin(7) can be defined as the stabilizer of the 4-form on $\mathbb{R}^8$

$$\Omega = e^0 \wedge \varphi + *\varphi,$$

where $\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$.

Definition

A *Spin*(7)-structure on a 8-manifold M^8 is given by a 4-form Ω with pointwise stabilizer isomorphic to *Spin*(7).

Remark

$Spin(7) \subset SO(8) \Rightarrow \Omega$ induces a metric g_Ω. Moreover, $*\Omega = \Omega$.

Proposition (Bryant; Fernández)

The following are equivalent:

(a) $\nabla^{LC}\Omega = 0$;

(a) $d\Omega = 0$;

(b) $Hol(g_\varphi)$ is isomorphic to a subgroup of $Spin(7)$.

A $Spin(7)$-structure satisfying $(a), (b)$ or (c) is called parallel.

Remark

Metrics induced by parallel $Spin(7)$-structures are Ricci-flat [Bonan].

The intrinsic torsion of a $Spin(7)$-structure

The intrinsic torsion of a $Spin(7)$-structure can be identified with

$$\nabla^{LC}\Omega \in T^* \otimes \mathfrak{spin}(7)^{\perp} \cong \Lambda^3 \cong \mathcal{W}_1 \oplus \mathcal{W}_2 \cong \Lambda^3_8 \oplus \Lambda^3_{48}$$

Some Fernández classes:

class	type	conditions
$\mathcal{W}_1$	balanced	$\theta = 0$
$\mathcal{W}_2$	locally conformal parallel	$d\Omega = \theta \wedge \Omega$

$\theta := -\frac{1}{7} * (*d\Omega \wedge \Omega)$ is the Lee form.

Remark

The Ricci tensor of the metric g_Ω can be expressed in relation to the intrinsic torsion [Karigiannis; Fowdar].

$Spin(7)$-structures on homogeneous spaces

Problem

Classify compact simply connected (almost effective) homogeneous manifolds M^8 with an invariant $Spin(7)$-structure.

• $M = G'/H'$ with G' a connected Lie group acting transitively and almost effectively.

• Ω is a G'-invariant 4-form stabilized by $Spin(7)$.

• Given $M = G'/H' \Rightarrow \exists$ a presentation $M = G/H$, where G is a compact connected simply connected, semisimple Lie group and H is a connected closed subgroup of G [Onishchik; Böhm]

The canonical presentation

Definition

A canonical presentation of a compact simply connected homogeneous space M is a presentation $M = G/H$, where G is a compact, connected simply connected semisimple Lie group and H is a connected closed subgroup of G.

- We restrict our attention to the canonical presentations.

- We use the classification of compact simply connected homogeneous spaces M^n of dim $n \leq 9$ [Klauss].

For the existence of *Spin*(7)-structures we use the following topological condition

Theorem (Lawson, Michelsohn)

A spin manifold M^8 admits a Spin(7)-structure if and only if $p_1^2(M^8) - 4p_2(M^8) + 8\chi(M^8) = 0$.

Remark

A compact simply connected Riemannian symmetric space cannot admit any invariant *Spin*(7)-structure!

Then we specify the canonical presentations for the compact simply connected almost effective non-symmetric homogeneous M^8.

Canonical presentations of M^8

	M^8	$\simeq G/H$
(1)	SU(3)	$\frac{SU(3)}{\{e\}}$
(2)	$\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$	$C_{k,\ell,m} := \frac{SU(2) \times SU(2) \times SU(2)}{U(1)_{k,\ell,m}}$, $k \geq l \geq m > 0, \gcd(k, \ell, m) = 1$,
(3)	$\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$	$\frac{SU(2) \times SU(2) \times SU(2)}{\Delta\, SU(2)} \times \frac{SU(2)}{U(1)}$
(4)	$\mathbb{S}^5_{V \oplus \mathbb{R}} \times \mathbb{S}^3$	$\frac{SU(3)}{SU(2)} \times SU(2)$
(5)	Sp(2)-full flag	$\frac{Sp(2)}{\mathrm{T}^2_{\mathrm{max}}}$
(6)	$\mathbb{F}^3 \times \mathbb{S}^2$	$\frac{SU(3)}{\mathrm{T}^2_{\mathrm{max}}} \times \frac{SU(2)}{U(1)}$
(7)	$\mathbb{CP}^3_{\mathfrak{m}_1 \oplus \mathfrak{m}_2} \times \mathbb{S}^2$	$\frac{Sp(2)}{Sp(1) \times U(1)} \times \frac{SU(2)}{U(1)}$
(8)	$\mathbb{S}^6_{\mathrm{irr}} \times \mathbb{S}^2$	$\frac{G_2}{SU(3)} \times \frac{SU(2)}{U(1)}$

Checking $p_1^2(M^8) - 4p_2(M^8) + 8\chi(M^8) = 0$

Proposition (Alekseevsky, F, Chrysikos, Raffero)

The canonical presentations of all compact simply connected non-symmetric almost effective homogeneous spaces admitting *Spin*(7)-structures are:

$$\frac{SU(3)}{\{e\}} \cong SU(3), \quad C_{k,\ell,m} \cong \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2,$$
$$\frac{SU(2) \times SU(2) \times SU(2)}{\Delta\, SU(2)} \times \frac{SU(2)}{U(1)} \cong \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2,$$
$$\frac{SU(3)}{SU(2)} \times SU(2) \cong \mathbb{S}^5_{V \oplus \mathbb{R}} \times \mathbb{S}^3 .$$

Remark

- $SU(3)$ admits a left-invariant $Spin(7)$-structure inducing the bi-invariant metric [Fernández].

- $C_{k,l,m}$ is a torus bundle over the the Kähler-Einstein $\left(\frac{\mathrm{SU}(2)}{U(1)}\right)^{\times 3}$ $\Rightarrow$ it is a non-Kähler C-space and it has an invariant Einstein metric.

- The Calabi-Eckmann $\frac{\mathrm{SU}(3)}{\mathrm{SU}(2)} \times \mathrm{SU}(2)$ is torus bundle over the Kähler-Einstein $\mathbb{CP}^2 \times \mathbb{CP}^1$ $\Rightarrow$ it is a non-Kähler C-space and it admits an invariant Einstein metric (which is unique!).

The three homogeneous spaces admit invariant Einstein metrics!

Examining when the isotropy subalgebra is a closed subalgebra of $\mathfrak{spin}(7)$

Theorem (Alekseevsky, F, Chrysikos, Raffero)

The canonical presentations of compact simply connected almost effective homogeneous spaces with an iinvariant Spin(7)-structure are exhausted by:

- $\frac{\mathrm{SU}(3)}{\{e\}}$
- *the infinite family* $C_{k,l,m}$, $k = l + m$,
- *the Calabi-Eckmann manifold* $\frac{\mathrm{SU}(3)}{\mathrm{SU}(2)} \times \mathrm{SU}(2)$.

The Characteristic connection

On every (M^8, Ω) $\exists!$ *Spin*(7)-connection ∇^c with totally skew-symmetric torsion: $T = -d^*\Omega - \frac{7}{6} * (\theta \wedge \Omega)$.

Theorem (Alekseevsky, F, Chrysikos, Raffero)

The spaces $C_{k,l,m}$ $(k > l > m > 0)$ and $\frac{\mathrm{SU}(3)}{\mathrm{SU}(2)} \times \mathrm{SU}(2)$ admit families of invariant Spin(7)-structures whose characteristic connection ∇^c has parallel torsion.

The *Spin*(7)-structures are of mixed type and induce naturally reductive metrics.

HAPPY BIRTHDAY, DMITRI!