SHORT TALK OF PROF. DMITRI ALEKSEEVSKY

Many sincere thanks for speakers for their very interesting talks and all participants for the contribution to the success of the conference. My deep appreciations and gratitude to the organizers of the Conference: Ioannis Chrysikos, Anton Galaev, Jan Slovák and Pavel Trojovský, for a big job and excellent organization of the online meeting. Special thanks for very clever choice of speakers.

Traditionally, mathematics are divided into pure and applied. V. Arnold disliked this division and insisted that there exists only one notion of mathematics. I agree with this. But according to motivation, one may divide mathematics into academic, or celestial mathematics and terrestrial mathematics. The motivation for developing celestial mathematics is a curiosity, while the development of terrestrial mathematics is stimulated by practical needs. I am glad that both parts are presented in this Conference. Celestial and terrestrial mathematics are complements to each other, and interrelations between them are strong and rather complicated. I would like to consider some examples:

(1) I. Kepler formulated his Laws, trying to understand "the harmony and music of celestial spheres". I. Newton used Kepler celestial laws to establish his fundamental Newton Laws, which became a basis of terrestrial physics and mathematics.
(2) E. Vinberg constructed in 60's the theory of *homogeneous convex cones* ([9, 10, 11]) guided by purely academic interest (looking for an answer on a question of É. Cartan). He showed that any such cone can be described as a cone of hermitian positive defined matrices with entries in some vector spaces (Vinberg matrix T-algebra). For any such cone, he introduced the so-called *characteristic function* φ, which is the density of the invariant measure (and may be considered as a generalization of the inverse determinant of a matrix). Vinberg associated with φ a canonical Riemannian metric, defined as the Hessian of $\log \varphi$. Essentially, at the same time, similar results had been obtained independently by J.-L. Koszul in his theory of homogeneous convex domains ([7]); hence the characteristic function φ is also known as Vinberg-Koszul function.[1]
(3) At the end of 80's, early of 90's, it was discovered by S. Cecotti, ([5]), B. de Wit and A. Van Proeyen ([12]), that for a special class of such rank 3 cone, the characteristic function defines a cubic polynomial h and the level set $h = const$ is a *very special real manifold*, which is the target space for mass multiplet in $5d$ Supergravity. Remarkably, such cones bijectively corresponds to Clifford modules. The dimensional reduction to $4d$ and $3d$ (called r-map and c-map, respectively), associates with a special Vinberg cone a *special Kähler manifold* and a special *homogeneous quaternionic Kähler manifold*, which are target spaces for multiplets for $d4$ and $d3$ Supergravity. The mathematical aspects of such celestial geometry was discussed in Vicente Cortés talk. One of the open problems of this celestial geometry is to calculate the entropy

[1] https://pdfs.semanticscholar.org/1ae6/be0f9f6af1a35d24fe49107bf2195166cc78.pdf

of the black hole, associated to a special Vinberg cone. It is solved only for a self-dual cone, see [4, 8] and the talk in Scholarpedia about Bekenstein-Hawking entropy.

(4) On the other hand, F. Barbaresco ([3]) has recently discovered that the Vinberg-Koszul theory may be interpreted in terms of Information Geometry, introduced by Fisher-Kulback-Chentsov-Amari, see [1]. Information geometry studies manifolds of probability measures (statistical manifold) equipped with a divergence (a sort of non symmetric distance) and an associated Fisher information metric. Vinberg cones provides a large family of the most interesting class of statistical manifolds–the so-called exponential family–which describes *generalized Wishart probability measures*, see [2], with the aforementioned Vinberg-Koszul canonical metric playing the role of the Fisher metric.

(5) The simplest Vinberg cone, i.e. the cone $\mathscr{P}^n$ of real positive defined symmetric matrices is identified with the manifold of (multivariate) Gauss measures. It plays a distinguished role in statistics and also in image analysis (in tomography and imaging of white matter in the brain–closely related with the talk by Remco Duits). The progress in tomography had been done, when researchers instead of the Euclidean metric started to use the Vinberg-Koszul canonical metric, which is the natural invariant metric of the symmetric space $\mathscr{P}^n = \mathsf{GL}^+(n, \mathbb{R})/\,\mathsf{SO}(n)$.

(6) Since the brain is a machine for information processing, information geometry is important to neuroscience and neurogeometry, which were the subjects of the talks by Alessandro Sarti and Giovanna Citti. In the framework of information geometry, K. Friston ([6]) introduced the *Principle of Minimization of Free Energy*, which states that

"the aim of the brain is to minimize the free energy."

Roughly speaking, the free energy is the difference between the probability distribution of environmental quantities that perceived by sense organs, and the distribution predicted by the brain.

(7) The Vinberg-Koszul characteristic function φ is important in Souriau theory of thermodynamics on Lie groups, and its generalization, proposed by F. Barbaresco ([3]). The *Koszul entropy* in this theory is defined as the Legendre transform of $\log \varphi$.

Above we saw how ideas by Vinberg and Koszul, motivated by celestial mathematic have strong and unexpected influence both on celestial and terrestrial mathematics. In 2019, there was the Montpellier Conference "*Foundation of Geometric Structures in Information*", dedicated to É. Cartan, J.-M.Souriau and J.-L.Koszul. Vasily Pestun from IHES, gave there a very interesting talk, where he explained that quantum field theory can be considered as an information geometry, where the *Zamolodchikov metric* plays the role of the Fisher metric. It is known that Misha Gromov is very skeptical about progress in geometry. In his talk, he said that there was not real advance in geometry since Euclid(!), and we need...

a new geometry.

(Before he was more optimistic and claimed that there was no progress after B. Riemann). But when I asked his impression about Vasily's talk, he said that it is exactly an example of the new geometry which we need.

It seems that Information Geometry (in the wide sense) is a discipline that unifies

Celestial Geometry and Terrestrial Geometry.

Then we may say that the Conference "*Geometry and Applications Online*", was devoted to different aspects of such a Generalized Information Geometry.

Thank you very much.

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