Kinematical G-structures and their intrinsic torsion

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Based on…

ON THE INTRINSIC TORSION OF SPACETIME STRUCTURES

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Dedicated to Dmitri Vladimirovich Alekseevsky on his eightieth birthday

ABSTRACT. We briefly review the notion of the intrinsic torsion of a G-structure and then go on to classify the intrinsic torsion of the G-structures associated with spacetimes: namely, galilean (or Newton–Cartan), carrollian, aristotelian and bargmannian. In the case of galilean structures, the intrinsic torsion classification agrees with the well-known classification into torsionless, twistless torsional and torsional Newton–Cartan geometries. In the case of carrollian structures, we find that intrinsic torsion allows us to classify Carroll manifolds into four classes, depending on the action of the Carroll vector field on the spatial metric, or equivalently in terms of the nature of the null hypersurfaces of a lorentzian manifold into which a carrollian geometry may embed. By a small refinement of the results for galilean and carrollian structures, we show that there are sixteen classes of aristotelian structures, which we characterise geometrically. Finally, the bulk of the paper is devoted to the case of bargmannian structures, where we find twenty-seven classes which we also characterise geometrically while simultaneously relating some of them to the galilean and carrollian structures.

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Introduction

"What are the possible geometric structures of space and time?"

Bacry + Lévy-Leblond (1968)

Classification of kinematical symmetries

Bacry + Nuyts (1986)

Classification of 4d kinematical Lie algebras

JMF (2017-18) JMF + Andrzejewski (2018)

Classification of kinematical Lie algebras (any dimension)

JMF + Prohazka (2018) JMF + Grassie + Prohazka (2019)

Classification of kinematical Klein geometries
(simply-connected, spatially isotropic, homogeneous spacetimes)

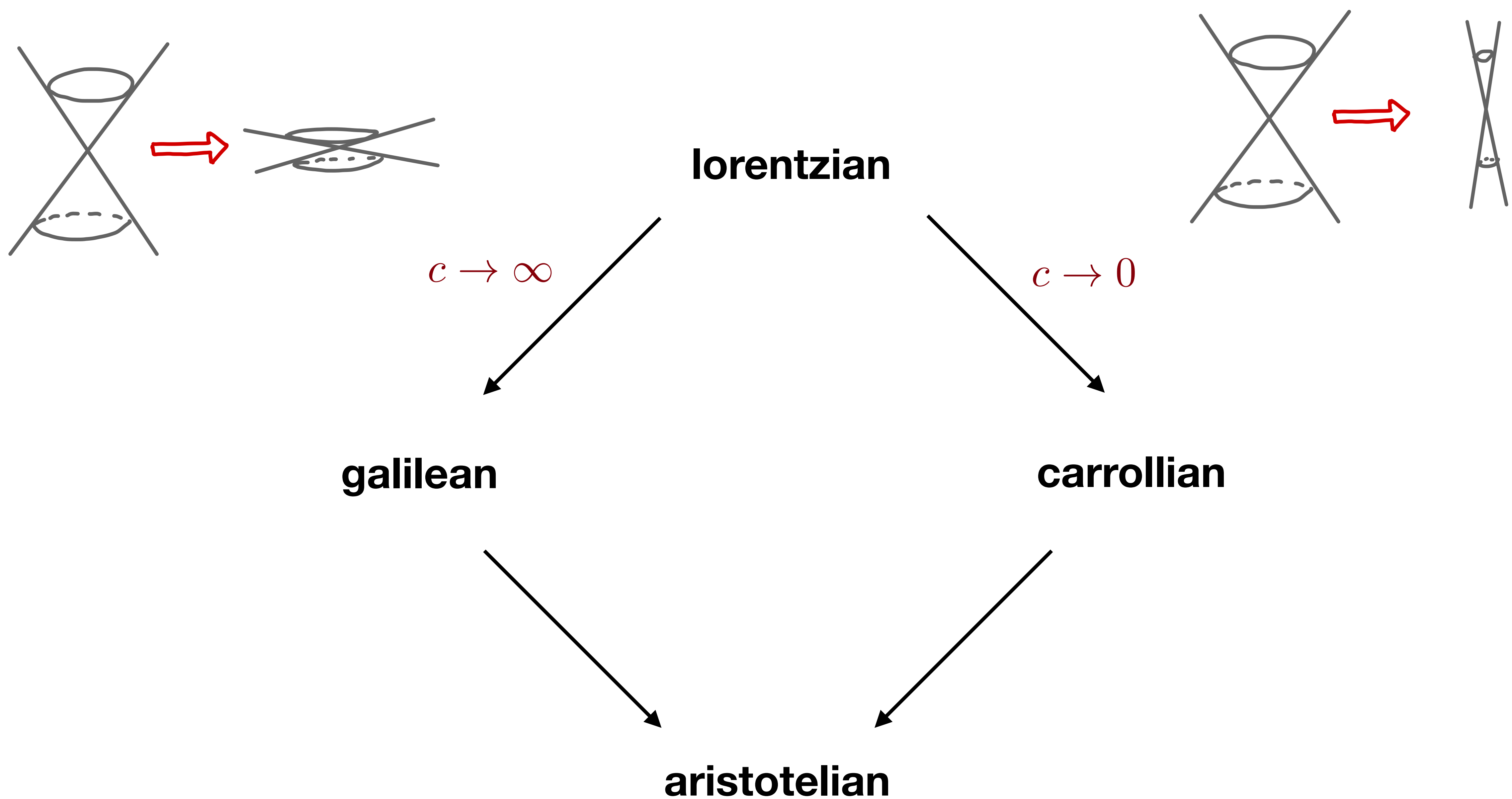
Families of spacetimes

- **lorentzian** (M, g)
- **galilean** (M, τ, γ) $\quad \tau \in \Omega^1(M) \quad \gamma \in \Gamma(\odot^2 TM)$

  $\gamma \geq 0 \quad \gamma(\tau, -) = 0$
- **carrollian** (M, ξ, h) $\quad \xi \in \mathscr{X}(M) \quad h \in \Gamma(\odot^2 T^* M)$

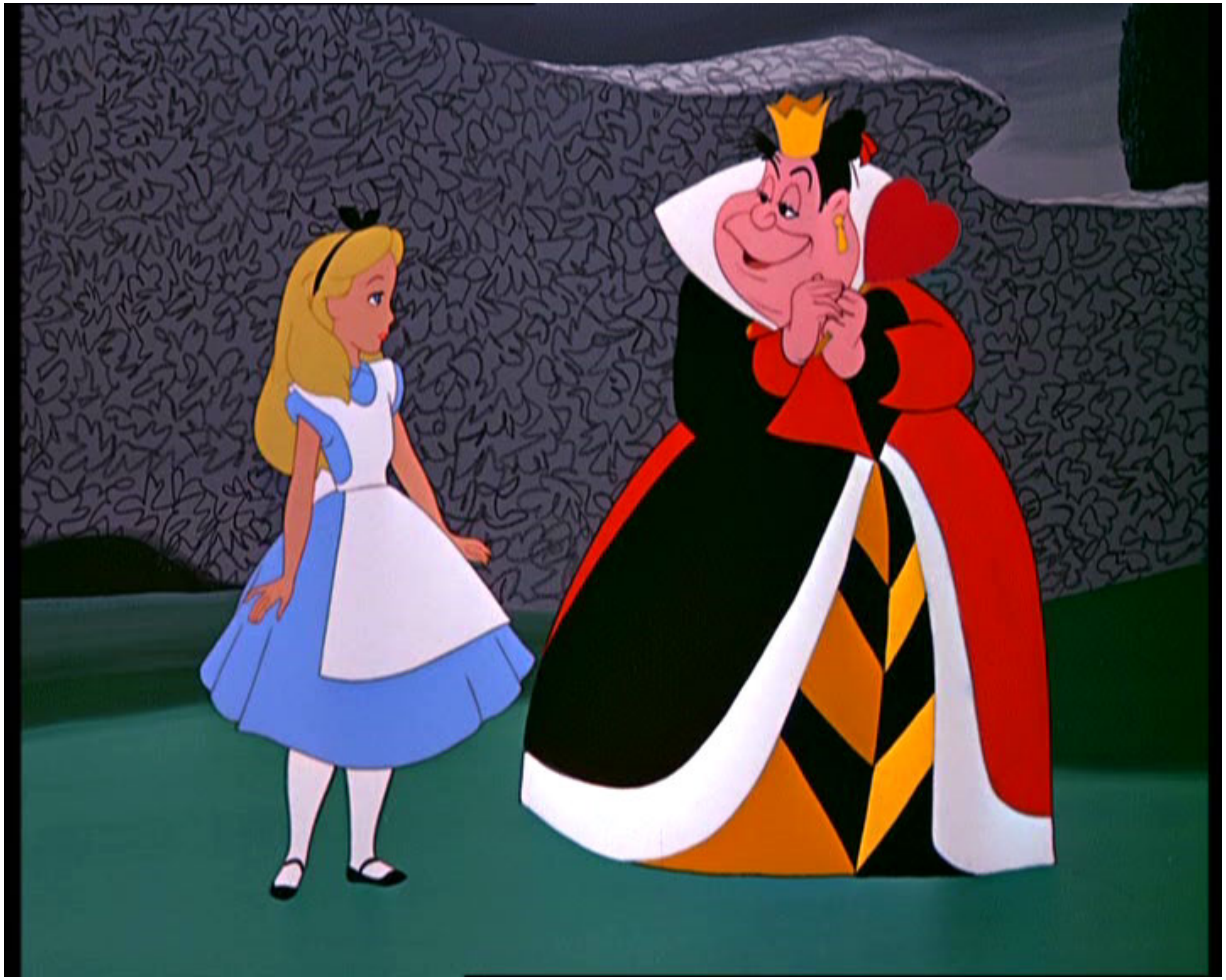
  $h \geq 0 \quad h(\xi, -) = 0$
- **aristotelian** $(M, \tau, \xi, \gamma, h)$ $\quad \tau(\xi) = 1$

Non- and ultra-relativistic limits



Why *carrollian*?

Dodgson (1865)
Lévy-Leblond (1965)



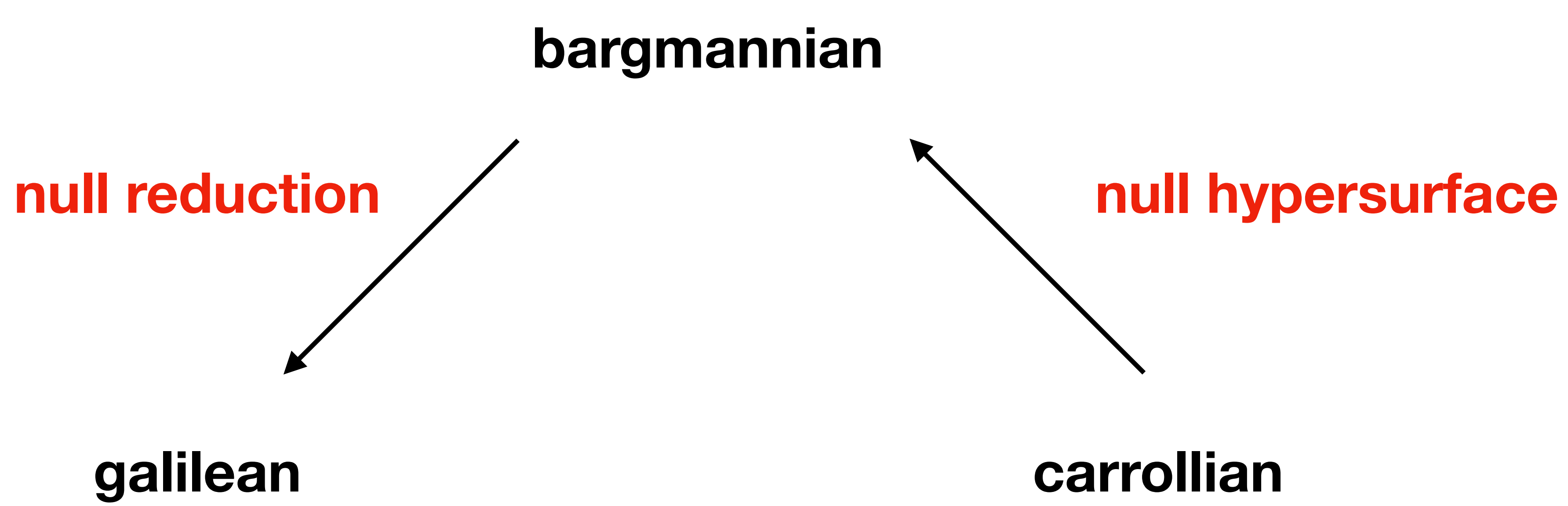
"My dear, here we must run as fast as we can, just to stay in place."

Bargmann geometry

Duval + Burdet + Künzle + Perrin (1985)
Duval + Gibbons + Horvathy + Zhang (2014)

(M, g) **lorentzian**

$\xi \in \mathcal{X}(M)$ **nowhere vanishing** $g(\xi, \xi) = 0$



Null reduction

Duval + Burdet + Künzle + Perrin (1985)
Julia + Nicolai (1994)

(M, g) **lorentzian**

Γ **one-dimensional connected Lie group** $\Gamma \curvearrowright M$ **isometrically**

$\xi \in \mathcal{X}(M)$ ***null* Killing vector field** $\pi : M \to \overline{M} := M/\Gamma$

$\tau \in \Omega^1(\overline{M})$ $\pi^*\tau = \xi^\flat$

$\gamma \in \Gamma(\odot^2 T\overline{M})$ $\pi^*\gamma(\alpha, \beta) = g((\pi^*\alpha)^\sharp, (\pi^*\beta)^\sharp)$

$\alpha, \beta \in \Omega^1(\overline{M})$

$\Longrightarrow$ $(\overline{M}, \tau, \gamma)$ **galilean**

Null hypersurfaces

Duval + Gibbons + Horvathy + Zhang (2014)
Hartong (2015)

(M, g) **lorentzian**

$\xi \in \mathcal{X}(M)$ **nowhere vanishing** $g(\xi, \xi) = 0$

Suppose $[\xi^\perp, \xi^\perp] \subset \xi^\perp$ $(\iff \xi^\flat \wedge d\xi^\flat = 0)$

$\xi \in \xi^\perp \implies$ **leaves of foliation are null hypersurfaces**

$i : N \hookrightarrow M$ $h = i^* g$ **degenerate**

$\implies (N, \xi, h)$ **carrollian**

G-structures

lorentzian $O(n-1,1) < \mathrm{GL}(n,\mathbb{R})$

galilean $$\left\{\begin{pmatrix} 1 & v^t \\ 0 & A \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R}) \;\middle|\; v \in \mathbb{R}^{n-1},\ A \in O(n-1)\right\} \cong O(n-1) \ltimes \mathbb{R}^{n-1}$$

carrollian $$\left\{\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R}) \;\middle|\; v \in \mathbb{R}^{n-1},\ A \in O(n-1)\right\} \cong O(n-1) \ltimes \mathbb{R}^{n-1}$$

aristotelian $$\left\{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R}) \;\middle|\; A \in O(n-1)\right\} \cong O(n-1)$$

bargmannian
$$\left\{\begin{pmatrix} 1 & -\frac{1}{2}v^t v & v^t \\ 0 & 1 & 0 \\ 0 & v & A \end{pmatrix} \in \mathrm{GL}(n+1,\mathbb{R}) \;\middle|\; v \in \mathbb{R}^{n-1},\ A \in O(n-1)\right\} \cong O(n-1) \ltimes \mathbb{R}^{n-1}$$

What are the intrinsic torsion classes of these G-structures?

Intrinsic torsion

M^n $\quad G < \mathrm{GL}(n, \mathbb{R})$ $\quad P \subset F(M)$ $\quad G$-structure

∇ affine connection adapted to P

$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ torsion

∇' another adapted affine connection $\kappa := \nabla' - \nabla \in \Omega^1(M; \mathrm{ad}\, P)$

$T^{\nabla'} - T^\nabla = \partial\kappa$ $\quad \partial : \Omega^1(M; \mathrm{ad}\, P) \to \Omega^2(M; TM)$

Spencer $\quad \partial\kappa(X, Y) := \kappa_X Y - \kappa_Y X$

$$\mathrm{coker}\, \partial = (TM \otimes \wedge^2 T^* M)/\partial(\mathrm{ad}\, P \otimes T^* M)$$

intrinsic torsion $\quad [T^\nabla] \in \Gamma(\mathrm{coker}\, \partial)$

Lorentzian structures

(M, g)

$$\ker \partial = \operatorname{coker} \partial = 0$$

There exists a unique torsion-free adapted connection

$=$

metric

(Fundamental theorem of riemannian geometry.)

Galilean structures

Künzle (1972)
Christensen + Hartong + Obers + Rollier (2013)

(M, τ, γ) $\qquad \tau \in \Omega^1(M)$ $\qquad \gamma \in \Gamma(\odot^2 TM)$

$$\ker \partial \cong \operatorname{coker} \partial \cong \wedge^2 T^* M$$

$$[T^\nabla] \mapsto d\tau$$

unique proper G-sub-bundle

$$\mathcal{G}_1 \subset \operatorname{coker} \partial$$

$\mathcal{G}_0 = 0$	$d\tau = 0$	**torsionless Newton—Cartan**
$\cap$		
$\mathcal{G}_1$	$\tau \wedge d\tau = 0$	**twistless torsional NC**
$\cap$		
$\mathcal{G}_2 = \operatorname{coker} \partial$	$\tau \wedge d\tau \neq 0$	**torsional NC**

$\dim M = 2, 5$ **are special**

Carrollian structures

(M, ξ, h) $\xi \in \mathcal{X}(M)$ $h \in \Gamma(\odot^2 T^* M)$

$$\ker \partial \cong \operatorname{coker} \partial \cong \odot^2 \operatorname{Ann} \xi = \langle h \rangle \oplus \odot_0^2 \operatorname{Ann} \xi$$

$$[T^\nabla] \mapsto \mathcal{L}_\xi h$$

$\mathcal{C}_0 = 0$ $\mathcal{L}_\xi h = 0$ **totally geodesic**

$\mathcal{C}_1 = \odot_0^2 \operatorname{Ann} \xi$ $\operatorname{div}_h \xi = 0$ **minimal**

$\mathcal{C}_2 = \langle h \rangle$ $\mathcal{L}_\xi h \propto h$ **totally umbilical**

$\mathcal{C}_3 = \odot^2 \operatorname{Ann} \xi$ **generic**

$\dim M = 2$ **is special**

Aristotelian structures

$(M, \xi, \tau, h, \gamma)$ $\quad \tau \in \Omega^1(M)$ $\quad \xi \in \mathcal{X}(M)$ $\quad \tau(\xi) = 1$

$\gamma \in \Gamma(\odot^2 TM)$ $\quad h \in \Gamma(\odot^2 T^* M)$

Sixteen classes $(\dim M \neq 2, 5)$

$$\begin{pmatrix} d\tau = 0 & \\ \tau \wedge d\tau = 0 & \\ \tau \wedge d\tau \neq 0 & \mathcal{L}_\xi \tau = 0 \\ \tau \wedge d\tau \neq 0 & \mathcal{L}_\xi \tau \neq 0 \end{pmatrix} \times \begin{pmatrix} \mathcal{L}_\xi h = 0 \\ \operatorname{div}_h \xi = 0 \\ \mathcal{L}_\xi h \propto h \\ \text{none of the above} \end{pmatrix}$$

Bargmannian structures

Duval + Burdet + Künzle + Perrin (1985)

(M, g, ξ) $\qquad g(\xi, \xi) = 0$

$G < O(n,1) \implies \ker \partial = 0$

$\operatorname{coker} \partial \cong \xi^{\perp} \otimes T^*M$

$[T^{\nabla}] \mapsto \nabla^g \xi$

∇^g **Levi-Civita connection**

$\dim M \neq 3, 6$ **27 classes**

$\dim M = 3$ **11 classes**

$\dim M = 6$ **47 classes**

Null distributions

(M, g, ξ) $\quad g(\xi, \xi) = 0 \quad$ $\xi^\perp \subset TM$

$L \subset \xi^\perp$ **line bundle spanned by** ξ $\quad E := \xi^\perp / L$

$$\begin{aligned} \xi^\perp &\to E \\ X &\mapsto \overline{X} \end{aligned}$$

E **has a riemannian structure**

$$h(\overline{X}, \overline{Y}) := g(X, Y)$$

In all but the generic case $[L, \xi^\perp] \subset \xi^\perp \quad (\iff \xi \text{ is geodetic})$

which guarantees the existence of a null Weingarten map

$$\begin{aligned} W : E &\to E \\ \overline{X} &\mapsto \overline{\nabla^g_X \xi} \end{aligned}$$

Null distributions

and a **null second fundamental form**

$$B(\overline{X}, \overline{Y}) := h(W(\overline{X}), \overline{Y}) = g(\nabla^g_X \xi, Y)$$

B is symmetric $\iff [\xi^\perp, \xi^\perp] \subset \xi^\perp$

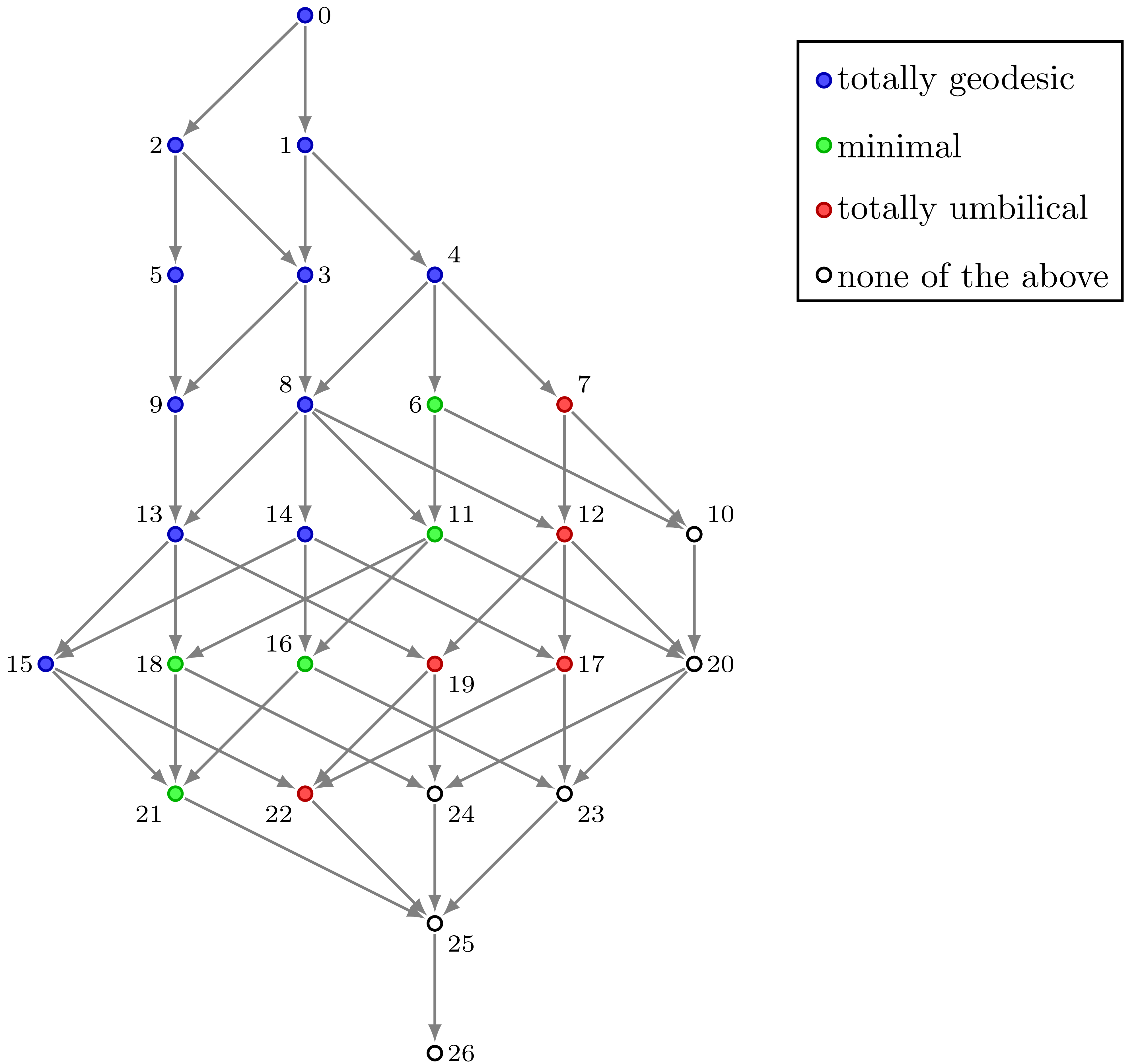
$$B_{\rm sym}(\overline{X}, \overline{Y}) := B(\overline{X}, \overline{Y}) + B(\overline{Y}, \overline{X})$$

$\xi^\perp$ is totally geodesic if $B_{\rm sym} = 0$

minimal if $\operatorname{tr} W = 0$

totally umbilical if $B_{\rm sym} \propto h$

generic otherwise.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totally geodesic
minimal
totally umbilical
none of the above

Outlook

- Are the inclusions strict? (cf. G_2 structures)
- Construct examples
- Higher invariants?
- Supersymmetric kinematical G-structures?

с Днем рожденья!