Dmitri's 80

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DmitriS80

The normal holonomy group of complex submanifolds

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Celebrating Dmitri's 80 birthday, September 2020

Abstract: This talk is going to be a survey of results, ideas and questions about the normal holonomy group of complex submanifolds of complex space forms.

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Subject: 1 year position in Hull
MIME-Version: 1.0

Dear Toni,
I will apply for a grant soon which give a possibility to invite a mathematician for 1 year for research (no teaching).
Are you interesting in such position. It may start in the middle of the next year . We have an idea to create a strong geometer group in Hull and there are possibilities to visit other universities in Europe and participate in conferences .

Best wishes,

Dmitri

International Conference CURVATURE IN GEOMETRY in honour of Professor Lieven Vanhecke, Lecce (Italy) was held June 11-14, 2003

Istituto Castelnuovo, Roma, 2003

1 Olmos' Normal Holonomy Theorem (NHT). [O90].

Theorem 3.1. *Let M^n be an immersed submanifold of a Riemannian manifold Q^N of constant curvature. Let $p \in M$ and let Φ^* be the restricted holonomy group of the normal connection at p. Then Φ^* is compact, there exists a unique (up to order) orthogonal decomposition of the normal space at p, $N(M)_p = \mathbf{V}_0 \oplus \cdots \oplus \mathbf{V}_k$, into Φ^*-invariant subspaces, and there exist $\Phi_0, \ldots, \Phi_k$ normal Lie subgroups of Φ^* such that:*

(i) $\Phi^* = \Phi_0 \times \cdots \times \Phi_k$ *(direct product).*
(ii) Φ_i *acts trivially on* $\mathbf{V}_j$ *if* $i \neq j$.
(iii) $\Phi_0 = \{1\}$ *and, if* $i \geq 1$, Φ_i *acts irreducibly on* $\mathbf{V}_i$ *as the isotropy representation of a simple Riemannian symmetric space.*

The above theorem plays a central role in the theory of isoparametric submanifolds or more in general in the theory of submanifolds with constant principal curvatures [BCO16].

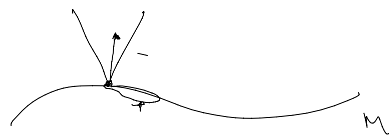
An important Theorem of Thorbergsson shows that a full an irreducible isoparametric submanifold of $\mathbb{R}^n$ of codimension greater than 2 is an orbit of an s-representation.

$\Phi_i \curvearrowright V_i$ s-representation.

G/K symmetric space

$K \curvearrowright T_{[K]} G/K$

$M \subset \mathbb{R}^n$



$T\mathbb{R}^N = TM \oplus \nu M$

$D = \nabla + \nabla^\perp$

①

② KE submanifolds $\mathbb{C}P^N$?

M^n complex $\hookrightarrow (\mathbb{C}P^N, \omega_{FS})$

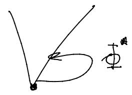
$(M^n, \omega_{FS}|_{TM})$

Conjecture: ARE HOMOGENEUS ?

2 NHT $\mathbb{C}^n$: The extrinsic De Rham splitting; [D00]

In [D00], which is part of my Ph.D. thesis I proved that an extrisinc de Rham Theorem. Roughly speaking, for a full complex submanifold $M \subset \mathbb{C}^n$ if $\Phi^{\perp}$ does not acts irreducible on $\nu_p(M)$ then M is an extrinsic product of two complex submanifolds.

Theorem 1.1 *A complex isometric full immersion of a simply connected complete Kähler manifold $f : M \to \mathbb{C}^N$ is irreducible, up a totally geodesic factor, if and only if the normal holonomy group acts irreducibly.*



$\boxed{M^n \subset \mathbb{C}^N}$

$$\begin{array}{ccc} M^n \neq & M_1 & \times & M_2 \\ & \cap & & \cap \\ & \mathbb{C}^{N_1} & \times & \mathbb{C}^{N_2} = \end{array}$$

extrinsic de Rham theorem.

3 NHT $\mathbb{C}\mathbb{P}^n$: work with D.V. Alekseevsky; [AD04]

Motivations ?

It turns out that Olmos NTH is not true if the submanifold is not full.

THEOREM 1. *Let $\mathbb{S}^n_c = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$ be the complex space form of holomorphic constant sectional curvature c and $M \subset \mathbb{S}^n_c$ be a Kähler submanifold. If the normal holonomy group $\mathrm{Hol}_p(\nabla^\perp) \subset \mathrm{SO}(N_p(M))$ of M at a point $p \in M$ acts irreducibly on the normal space $N_p(M)$ then $\mathrm{Hol}_p(\nabla^\perp)$ is linearly isomorphic to the isotropy group of an irreducible Hermitian symmetric space, that is, one of the groups in Table 1. In particular, this is true if $M \subset \mathbb{C}^n$ is a locally irreducible Kähler submanifold of $\mathbb{C}^n$.*

TABLE 1. *Isotropy representations $K \hookrightarrow \mathrm{SO}(V)$ of compact irreducible Hermitian symmetric spaces G/K.*

G/K	K	V
$\mathrm{Gr}_p(\mathbb{C}^{p+q}) := \mathrm{SU}(p+q)/S(U(p)\times U(q))$	$S(U(p)\times U(q))$	$\mathbb{C}^p \otimes \mathbb{C}^q$
$\mathrm{SO}(2n)/U(n)$	$U(n)$	$\Lambda^2(\mathbb{C}^n)$
$\mathrm{Gr}_2(\mathbb{R}^{n+2}) := \mathrm{SO}(n+2)/\mathrm{SO}(2)\times\mathrm{SO}(n)$	$\mathrm{SO}(2)\times\mathrm{SO}(n)$	$\mathbb{R}^2 \otimes \mathbb{R}^n$
$\mathrm{Sp}(n)/U(n)$	$U(n)$	$S^2\mathbb{C}^n$
$E_6/T^1\cdot\mathrm{Spin}_{10}$	$T^1\cdot\mathrm{Spin}_{10}$	$\mathbb{C}^{16}$
$E_7/T^1\cdot E_6$	$T^1\cdot E_6$	$\mathbb{C}^{27}$

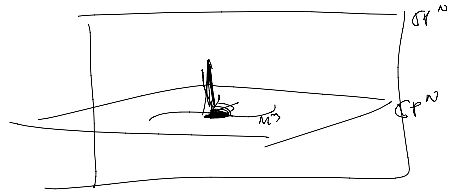
$\mathbb{C}P^1 \subset \mathbb{C}P^3$

totally geodesic $\quad 2$

GAUSS – Codazzi – Ricci ,

$\Phi^* \curvearrowright \mathbb{C}^2 = \nu_p \mathbb{C}P^1$

$SO(2) \quad e^{i\theta}$ is NOT S-repres.



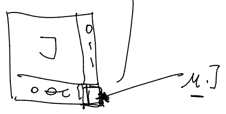
$M^n \subset \mathbb{C}P^N \subset \mathbb{C}P^{N+1}$ full

isotr. Rep. Hermitian

ν

$J \in \Phi^* \quad \forall M^n$

$\mathbb{C}P^{N+1} = \mathbb{C} + \nu_{M^n}\mathbb{C}P^N$



THEOREM 12. *Let $(M^m, g) \subset \mathbb{C}P^n$ be a non-full and non-totally geodesic Kähler–Einstein submanifold with Ricci tensor $\mathrm{Ric}_M = k\cdot g$. Let $\mathbb{C}P^{\overline{m}} \subset \mathbb{C}P^n$ be the totally geodesic Kähler submanifold of $\mathbb{C}P^n$ such that M is full in $\mathbb{C}P^{\overline{m}}$. Then,*

$$\mu = \frac{\overline{m} - m}{\overline{m} + 1 - k/2}$$

where μ is the invariant of Definition 1.

rational Hollin → compactness Φ^*

THEOREM 13. *Let M^m be a Kähler submanifold of the space form $\mathbb{S}^n_c = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$ with $c = 4, 0, -4$ respectively. If $\dim(\mathrm{hol}(\nabla^\perp)) = 1$ then one of the following holds:*

(i) *$c \neq 0$ and M is a complex hypersurface;*
(ii) *$c \neq 0$ and M is a totally geodesic submanifold;*
(iii) *$c = 0$ and M is a complex hypersurface in a complex affine subspace of $\mathbb{C}^n$;*
(iv) *$c = 4$ and M^m is congruent to an open subset of a non-full complex quadric $Q^m := \{[z_0 : \ldots : z_{m+1}] \in \mathbb{C}P^{m+1} \mid z_0^2 + \ldots + z_{m+1}^2 = 0\} \subset \mathbb{C}P^n$, with $m + 1 < n$.*

$\boxed{SO(2)\times SO(2) = \Phi^*}$

$M^2 \subset \mathbb{C}P^2 \subset \mathbb{C}P^3$

$M^1 \subset \mathbb{C}P^3$

$\Phi^* \subset \underbrace{SO(2)\times SO(2)}_{2}$

$0, 1 \quad 0$

$\Phi^* = (1)$

4 NHT $\mathbb{CP}^n$: works with S. Console and C. Olmos; [CD09, CDO11]

Olmos–Heintze

{ isoparametric submanifolds

$\mathbb{R}^n$, S^n, H^n

Olmos.

11 simetric s-represent.

Parallel submanifolds of complex projective space 3

Table 1 Symmetric complex submanifolds $M \subset \mathbb{P}(T_{[K]}G/K)$

Hermitian symmetric space G/K	M as complex K-orbit	Normal holonomy	Remarks
$\frac{E_7}{T^1 \cdot E_6}$	$\frac{E_6}{T^1 \cdot Spin_{10}}$	$\frac{SO(12)}{T^1 \cdot SO(10)}$	
$\frac{E_6}{T^1 \cdot Spin_{10}}$	$\frac{SO(10)}{U(5)}$	$\frac{U(6)}{U(5)}$	
$\frac{Sp(n+1)}{U(n+1)}$	$\mathbb{C}P^n$	$\frac{Sp(n)}{U(n)}$	Veronese
$Gr_2^+(\mathbb{R}^{n+2}) := \frac{SO(n+2)}{T^1 \cdot SO(n)}$	$Gr_2^+(\mathbb{R}^n)$	$\frac{U(2)}{U(1)}$	Quadrics
$\frac{SO(2n)}{U(n)}$	$Gr_2(\mathbb{C}^n)$	$\frac{SO(2(n-2))}{U(n-2)}$	Plücker
$Gr_a(\mathbb{C}^{a+b}) := \frac{SU(a+b)}{S(U(a) \times U(b))}$	$\mathbb{C}P^{a-1} \times \mathbb{C}P^{b-1}$	$\frac{SU(a+b-2)}{S(U(a-1) \times U(b-1))}$	Segre

The space in the third column is the Hermitian symmetric space whose isotropy representation gives the normal holonomy action

i Hermitian S

G/K

$K \curvearrowright T_{[K]} G/K$

$K \curvearrowright \mathbb{P}\, T\, G/K$

← Realization Problem!

Note: the two exceptional spaces and the quadrics Q_n, $n \notin \{1, 2, 3, 4, 6, 10\}$ are not in the third column

4.1 An extrinsic Berger type theorem; [CDO11]

Theorem 2 *The normal holonomy group of a complete irreducible and full immersed complex submanifold of $\mathbb{C}^n$ acts transitively on the unit sphere of the normal space. Indeed, $\Phi^\perp = U(k)$, where k is the codimension of the submanifold.*

Theorem 1 *Let M be a full and complete complex projective submanifold of $\mathbb{C}P^n$. Then the following are equivalent:*

(1) *The normal holonomy is not transitive on the unit sphere of the normal space (i.e., different from $U(k)$, $k = codim(M)$, since it is an s-representation).*

(2) *M is the complex orbit, in the complex projective space, of the isotropy representation of an irreducible Hermitian symmetric space of rank greater or equal to 3.*

5 NHT $\mathbb{CP}^n$: work with F. Vittone. [DV17]

Theorem 1. *Let $M \subset \mathbb{C}^n$ be a full and irreducible complex submanifold (non necessarily complete w.r.t. the induced metric of $\mathbb{C}^n$). Let $Hol^*(M, \nabla^\perp)$ be the restricted normal holonomy group of M. If the action of $Hol^*(M, \nabla^\perp)$ is non-transitive on the unit sphere of the normal space then there exists an irreducible bounded symmetric domain $D \subset \mathbb{C}^n$ (realized as a circled domain) such that M is an open subset of the smooth part of the Mok's characteristic cone $\mathcal{CS}^j(D)$ for $1 \le j < rank(D) - 1$.*

Conversely, for any irreducible bounded symmetric domain $D \subset \mathbb{C}^n$, the restricted normal holonomy group of an open subset of the smooth part of the cone $\mathcal{CS}^j(D)$ for $1 \le j < rank(D) - 1$ acts irreducibly but non-transitively on the unit sphere of each normal space.

DEFINITION 1

Let $1 \le k \le r(\Omega)$ and let $\mathcal{S}_{k,x}$ denote $\{[\xi]: \xi \in T_x(\Omega)$ and $1 \le r(\xi) \le k\}$. We call $\mathcal{S}_{k,x}(\Omega) \subset \mathbb{P}T_x(X)$ the k–th characteristic projective subvariety at $x \in \Omega$. The union $\mathcal{S}_k(\Omega) := \cup_{x\in\Omega} \mathcal{S}_{k,x} \subset \mathbb{P}T(\Omega)$ is called the k–th characteristic bundle over Ω. The quotient $\mathcal{S}_k(\Omega)/\Gamma$ (noting that $\mathcal{S}_k(\Omega)$ is invariant under the standard action of Γ), written $\mathcal{S}_k(X)$, is called the k–th characteristic bundle over X.

Algebr

[Mok89, page 252]

$G/K \to \mathbb{V} = T\,G/K$

K

$\xi \in \mathbb{V}$

$r(\xi)$

$M^m \subset \mathbb{C}P^N = P(\mathbb{C}^{N+1})$

$\mathbb{C}^{N+1} = \mathbb{C}^a \times \mathbb{C}^b$

Cone X –

MAIN obstacle

Olmo's Proof Normal theorem

$\mathbb{C}P^N$ or $\mathbb{C}H^n$ Ricci.

S^N or H^m Euclidean

Dmitri –

$$\boxed{\langle R^\perp_{XY}\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle} + \langle\ \rangle$$

$R^\perp$ normal curvature Tensor of $\nabla^\perp$

A_ξ is shape operator in $\xi \in \nu$

6 Two open problems.

For complex submanifolds of $\mathbb{CH}^n$ the conjecture is that if the normal holonomy is irreducible but non transitive then must be the full unitary group of the normal space.

We remark that there are examples of full complex submanifolds of $\mathbb{CH}^n$ whose normal holonomy is not irreducible.

Is the invariant μ a rational number ? or equivalently is the normal holonomy a compact Lie group?

$\mathbb{C}H^n$? $\mathbb{C}^{N+1}$ Lorentz

$\mathbb{C}P^n$ $\mathbb{C}^{N+1}$ — Euclidean

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