# Quaternionic Kähler manifolds of cohomogeneity one

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# I. Quaternionic Kähler manifolds

### Reminder

A Kähler manifold is a Riemannian manifold $(M, g)$ endowed with a parallel, skew-symmetric (almost) complex structure J.

### Definition

A quaternionic Kähler manifold $(M, g, Q)$ of dimension $> 4$ is a Riemannian manifold $(M, g)$ of dimension $> 4$ endowed with a parallel, skew-symmetric (almost) quaternionic structure $Q$.

### In terms of holonomy:

Kähler $\iff \mathrm{Hol} \subset \mathrm{U}(n)$.
Quaternionic Kähler dim $> 4 \iff \mathrm{Hol} \subset \mathrm{Sp}(n)\mathrm{Sp}(1)$, $n > 1$.

### Simplest examples

$\mathbb{C}P^n$ has $\mathrm{Hol} = \mathrm{U}(n)$.
$\mathbb{H}P^n$ has $\mathrm{Hol} = \mathrm{Sp}(n)\mathrm{Sp}(1)$.

# Wolf and Alekseevsky spaces

## Theorem (Berger ‘55, Alekseevsky ‘68)

Let $(M, g)$ be a simply connected non locally symmetric Riemannian manifold with irreducible holonomy group. Then

$$\mathrm{Hol} \in \{\mathrm{SO}(n),\ \mathrm{U}(n),\ \underbrace{\mathrm{Sp}(n)\mathrm{Sp}(1),\ \mathrm{SU}(n),\ \mathrm{Sp}(n),\ \mathrm{G}_2,\ \mathrm{Spin}(7)}_{\Longrightarrow \text{Einstein}}\}.$$

## Symmetric QK manifolds

1. Those of compact type (Wolf spaces) are in natural bijection with the complex simple Lie algebras (Wolf ‘68).
2. The noncompact duals of the Wolf spaces are also QK.

## Alekseevsky ‘75

Classified QK manifolds with a simply transitive completely solvable group of isometries (Alekseevsky spaces) and found the first non locally symmetric QK manifolds.

# State of the art

## Open problem

No compact, non locally symmetric quaternionic Kähler manifold of $\mathrm{scal} \neq 0$ known.

Compare: for all the other groups of Einstein type from Berger's list, compact examples have been constructed by Yau '78 ($\mathrm{SU}(n)$), Beauville '83 ($\mathrm{Sp}(n)$), and Joyce '94 ($\mathrm{G}_2$, $\mathrm{Spin}(7)$).

## LeBrun-Salamon Conjecture '94

A complete quaternionic Kähler manifold of $\mathrm{scal} > 0$ is a Wolf space.

## Evidence

1. True for $\dim M \leq 16$ (Hitchin '81, Friedrich-Kurke '82, Poon-Salamon '91, Buczyński-Wiśniewski-Weber '20).
2. Finiteness theorem (LeBrun-Salamon '94).
3. $\nexists$ positive QK manifolds of cohomogeneity one (Dancer-Swann '99, Podestà-Verdiani '00).

# State of the art continued

## Bagger-Witten '83

Quaternionic Kähler manifolds of $\mathrm{scal} < 0$ are related to supergravity.

## Consequences

This fact has lead to a number of physics inspired constructions:

1. r-map (de Wit-Van Proeyen '92)
2. c-map (Ferrara-Sabharwal '90)
3. quantum corrections in string theory:
   3.1 perturbative (Robles-Llana-Saueressig-Vandoren '06)
   3.2 instanton corrections (see Alexandrov-Manschot-Persson-Pioline arXiv:1304.0766 for a survey)

## Plan for this talk

... explain how to obtain complete QK manifolds, including cohomogeneity one examples.

# II. The q-map

### Fact

The q-map, i.e. r-map followed by c-map, is a construction which associates a quaternionic Kähler manifold with every projective special real manifold.

### Definition

A projective special real (PSR) manifold is a hypersurface $\mathcal{H} \subset \mathbb{R}^n$ s.t. $\exists$ homog. cubic polynomial $h$ on $\mathbb{R}^n$ s.t.

i) $h = 1$ on $\mathcal{H}$ and

ii) $\partial^2 h$ is negative definite on $T\mathcal{H}$.

Then $\iota : \mathcal{H} \hookrightarrow \mathbb{R}^n$ is endowed with the Riemannian metric

$$g_{\mathcal{H}} = -\frac{1}{3}\iota^*\partial^2 h.$$

# The q-map: some global results

**Theorem (C.-Han-Mohaupt '12)**

The q-map preserves completeness.

**Theorem (C.-Nardman-Suhr '16)**

A PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^n$ is complete iff $\mathcal{H} \subset \mathbb{R}^n$ is closed.

**Corollary (C.-Nardman-Suhr '16)**

Let $\mathcal{H}$ be a locally strictly convex component of the level set $\{h = 1\}$ of a homogeneous cubic polynomial $h$ on $\mathbb{R}^n$. Then $\mathcal{H}$ determines a complete QK manifold diffeomorphic to $\mathbb{R}^{4n+4}$.

# III. Automorphisms

## Proposition (r-map)

Let $\mathcal{H} \subset \mathbb{R}^n$ be a projective special real manifold and $\bar{M}$ the corresponding projective special Kähler (PSK) domain.
The action of $\mathrm{Aut}(\mathcal{H})$ extends to an action of $\mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^n) := (\mathbb{R}^{>0} \times \mathrm{Aut}(\mathcal{H})) \ltimes \mathbb{R}^n$ on $\bar{M}$ by automorphisms.

## Proof (sketch)

As a Kähler manifold $\bar{M}$ can be described as follows.

- Complex structure:

$$\bar{M} = U \times \mathbb{R}^n = \mathbb{R}^n + iU \subset \mathbb{C}^n,$$

  where $U = \mathbb{R}^{>0} \cdot \mathcal{H} \subset \mathbb{R}^n$.
- Metric: $g_{\bar{M}} = \frac{3}{4} \sum g_{ab}(dx^a dx^b + dy^a dy^b)$, where

$$g_U = -\frac{1}{3}\partial^2 \log h = \sum g_{ab} dx^a dx^b.$$

# Automorphisms: r-map continued

## Proof continued

The fact that $\bar{M}$ is a PSK domain means that $\exists$ holomorphic function $F : M_F \to \mathbb{C}$ homog. of deg. 2 on $\mathbb{C}^*$-invariant domain $M_F \subset \mathbb{C}^{n+1} \setminus \{0\}$ s.t. $\bar{M}$ can be realised as the image of the Lagrangian cone $M := \mathrm{im}(dF : M_F \to T^*\mathbb{C}^{n+1} = V)$ under the projection $\pi : V \setminus \{0\} \to P(V)$, $V = (\mathbb{C}^{2n+2}, \Omega = \sum dz^j \wedge dw_j)$ with the metric induced by $g_V = \mathrm{Re}(i\Omega(\cdot, \bar{\cdot}))$.

In fact, here $F = \frac{h(z^1,\ldots,z^n)}{z^0}$, $M_F = \{z^0 \cdot (1, p) \mid p \in \bar{M},\ z^0 \in \mathbb{C}^*\}$, $\bar{M} \subset \mathbb{C}^n \subset \mathbb{C}P^n$ is identified with $P(M_F) \subset \mathbb{C}P^n$ and $P(M_F)$ with $P(M) \subset P(V) = \mathbb{C}P^{2n+1}$. So $\bar{M} \cong P(M)$ is a PSK domain.

Now we can construct embedding $\varphi_h : \mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^n) \to \mathrm{Aut}(M) \subset \mathrm{Sp}(\mathbb{R}^{2n+2})$ (inducing embedding into $\mathrm{Aut}(\bar{M})$) as follows ...

# Automorphisms: r-map continued

1. Restriction of $\varphi_h$ to $\mathrm{Aut}(\mathcal{H}) \subset \mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^n)$ defined by

$$\mathrm{Aut}(\mathcal{H}) \subset \mathrm{GL}(n,\mathbb{R}) \subset \mathrm{GL}(n+1,\mathbb{R}) \subset \mathrm{Sp}\left(\mathbb{R}^{2n+2}\right).$$

2. Restriction to $\mathbb{R}^{>0}$ given by

$$\mathbb{R}^{>0} \ni \lambda \mapsto \mathrm{diag}(\lambda^{-\frac{3}{2}}, \lambda^{-\frac{1}{2}}\mathbf{1}) \in \mathrm{GL}(n+1,\mathbb{R}) \subset \mathrm{Sp}\left(\mathbb{R}^{2n+2}\right).$$

3. $\varphi_h|_{\mathbb{R}^n} : \mathbb{R}^n \to \mathrm{Sp}\left(\mathbb{R}^{2n+2}\right)$ given by

$$\varphi_h(v) = \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline v & \mathbf{1} & 0 & 0 \\ \hline -H(v,v,v) & -3H(v,v,\cdot) & 1 & -v^t \\ \hline 3H(v,v,\cdot)^t & 6H_v & 0 & \mathbf{1} \end{array}\right),$$

$H \in S^3(\mathbb{R}^n)^*$ defined by $H(v,v,v) = h(v)$, $v \in \mathbb{R}^n$, and $H_v : \mathbb{R}^n \to \mathbb{R}^n$, $z \mapsto H(v,z,\cdot)^t$. □

# Automorphisms: c-map

### Proposition (c-map)

Let $M \to \bar{M}$ be a conical affine special Kähler domain and $(\bar{N} = \bar{M} \times G, g_{\bar{N}})$ the corresponding QK manifold. The action of $\mathrm{Aut}(M)$ on $\bar{M}$ extends to an action of

$$\mathrm{Aut}(M) \ltimes G \subset \mathrm{Sp}(\mathbb{R}^{2n+2}) \ltimes G$$

on $\bar{N}$ by isometries. Here $G = \mathrm{Iwa}(\mathrm{SU}(1, n+2))$.

# Automorphisms: q-map

**Theorem (q-map) [C.-Dyckmanns-Jüngling-Lindemann '20]**

If $\mathrm{Aut}(\mathcal{H})$ acts with cohomogeneity $k \in \mathbb{N}_0$ on $\mathcal{H}$, then

$$\mathrm{Isom}_{\mathcal{H}}(\bar{N}) := \varphi_h(\mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^n)) \ltimes G \subset \mathrm{Aut}(M) \ltimes G \subset \mathrm{Isom}(\bar{N})$$

acts with cohomogeneity $k$ on $\bar{N}$ and $\mathrm{Isom}(\bar{N})$ with cohomogeneity $\leq k$ on $\bar{N}$. In particular we always have

$$\mathrm{coh}(\mathrm{Isom}(\bar{N})) \leq \dim \mathcal{H} = \frac{\dim \bar{N}}{4} - 2.$$

# IV. A series of cohomogeneity 1 examples

Theorem [CDJL]

Consider $h(x) = x_1(x_1^2 - \sum_{i=2}^n x_i^2)$, $n \geq 2$. Then $\mathcal{H} = \{h(x) = 1,\ x_1 > 0\} \subset \mathbb{R}^n$ is a complete PSR manifold and the corresponding complete quaternionic Kähler manifold $\bar{N}$ is of cohomogeneity one.

Remark:

In this case, $\mathrm{Isom}_{\mathcal{H}}(\bar{N}) = ((\mathbb{R}^{>0} \times \mathrm{O}(n-1)) \ltimes \mathbb{R}^n) \ltimes_{\varphi_h} G$.

# V. HK/QK-correspondence and 1-loop deformation

## Facts

1. $\exists$ extension of Haydy's HK/QK-correspondence for indefinite HK manifolds, which applies to the manifolds $N$ obtained by the HK-version of the c-map and yields a family of QK metrics $g^c_{\bar{N}}$ [Alekseevsky-C.–Mohaupt '13].
2. The metric $g^0_{\bar{N}}$ coincides with $g_{\bar{N}}$ obtained by the c-map [Alekseevsky-C.–Dyckmanns-Mohaupt '15]:

$$\begin{array}{ccc} M & \xrightarrow{c} & N \\ {\scriptstyle \mathbb{C}^*}\downarrow & & \downarrow{\scriptstyle \text{HK/QK}} \\ \bar{M} & \xrightarrow{\bar{c}} & \bar{N} \end{array}$$

3. This yields a geometric proof that the manifold $(\bar{N}, g_{\bar{N}})$ and a 1-parameter deformation thereof is QK.
4. The deformed QK manifold $(\bar{N}, g^c_{\bar{N}})$, $c > 0$, is complete if the initial PSK manifold $\bar{M}$ is complete and of regular boundary behavior or in the image of the r-map [C.-Dyckmanns-Suhr '17].

# VI. Cohomogeneity 1 by 1-loop deformation of homogeneous spaces

**Theorem 1 (C.-Saha-Thung arXiv:2001.10026)**

Let $N$ be a possibly indefinite HK manifold and $f_Z$ a function which satisfies the assumptions of the HK/QK-correspondence. Then there is a central extension of $\mathfrak{aut}(N, f_Z)$ which acts by Killing vector fields on $(\bar{N}, g^c_{\bar{N}})$.

**Theorem 2 (C.-Saha-Thung 2001.10026)**

Let $\bar{M}$ be a homogeneous PSK domain of dimension $2n$. Then the complete QK manifold $(\bar{N}, g^c_{\bar{N}})$, $c > 0$, obtained by 1-loop deformation of the c-map manifold $(\bar{N}, g^0_{\bar{N}})$, is of cohomogeneity $\leq 1$.

**Remark**

Above manifolds $(\bar{N}, g^0_{\bar{N}})$ exhaust all known examples of homogeneous QK manifolds of $\mathrm{scal} < 0$ with exception of $\mathbb{H}H^n$.

# Cohomogeneity 1 by 1-loop deformation of homogeneous spaces

## Conjecture

The 1-loop deformation of a homogeneous QK manifold is always of cohomogeneity 1.

## Theorem 3 (C.-Saha-Thung 2001.10032)

Applying the above construction to $\bar{M} = \mathbb{C}H^n$ we obtain a deformation $(\bar{N}, g^c_{\bar{N}})$, $c > 0$, of the symmetric QK manifold

$$(\bar{N}, g^0_{\bar{N}}) = \frac{\mathrm{SU}(2, n+1)}{\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(n+1))}$$

by complete QK manifolds of cohomogeneity 1.

## Theorem 4 (follows from Theorem 2 and [CDS])

The 1-loop deformation of the symmetric QK manifold $\mathrm{G}_{2(2)}/\mathrm{SO}(4)$ is of cohomogeneity 1.

# VII. How to prove cohomogeneity 1

## Cohomogeneity $\leq 1$

1. Infinitesimal automorphisms of CASK manifold $M$ lift to automorphism of $(N = T^*M, f_Z)$, where $N$ is endowed with the semi-flat HK structure and $Z$ is the $\nabla$-horizontal lift of $J\xi$.
2. Applying Theorem 1 we show that 1-loop deformation $(\bar{N}, g^c_{\bar{N}})$, $c > 0$ of c-map manifold $\bar{N}$ associated with $\bar{M}$ admits a group of isometries inducing the transitive action on base of $\bar{N} = \bar{M} \times G \to \bar{M}$.
3. On the other hand there is always $\mathrm{Heis}_{2n+3} \subset G$ acting freely by isometries and transversally to the above action.
4. This yields an isometric cohomogeneity one action of $\mathrm{Aut}(\bar{M}) \ltimes \mathrm{Heis}_{2n+3}$ on $(\bar{N}, g^c_{\bar{N}})$, $c > 0$.

## Cohomogeneity $\geq 1$

For this we prove a general structure result for curvature tensor under HK/QK-correspondence and specialize it to above series.

# VIII. Curvature tensor under HK/QK-correspondence

- Let $(N, g_N, f_Z)$ be input data for the HK/QK-correspondence and $(\bar{N}, g^c_{\bar{N}})$ the resulting family of QK manifolds.
- The metrics are conveniently related , by a double fibration

$$N \longleftarrow (P, \eta) \longrightarrow \bar{N} \quad \text{[ACDM]}.$$

- Using the terminology of [Macía and Swann '15], $g^c_{\bar{N}}$ is $\eta$-related to an elementary deformation

$$\begin{aligned} g_H &= \frac{1}{f_Z} g_N + \frac{1}{f_Z^2}((\iota_Z g_N)^2 + \sum_{k=1}^{3} (\iota_Z \omega_k)^2) \\ &= \frac{1}{f_Z}\, g_N|_{(\mathbb{H}Z)^\perp} + \frac{f_H}{f_Z^2}\, g_N|_{\mathbb{H}Z} \,. \end{aligned}$$

of $g_N$, where $f_H = f_Z + g_N(Z, Z)$.

# Curvature tensor under HK/QK-correspondence

## Theorem ([CST 2001.10032])

The curvature $(4,0)$-tensor $R^c$ of the QK metric $g^c_N$ is $\eta$-related to the tensor

$$\frac{1}{f_Z}R + \frac{1}{8}\left(g_H \owedge g_H + \sum_{k=1}^{3} g_H(I_k\cdot,\cdot) \varobar g_H(I_k\cdot,\cdot)\right)$$
$$-\frac{1}{8f_Z f_H}\left(\omega_H \varobar \omega_H + \sum_{k=1}^{3} \omega_H(I_k\cdot,\cdot) \owedge \omega_H(I_k\cdot,\cdot)\right),$$

where $R$ denotes the curvature $(4,0)$-tensor of the HK metric $g_N$, $I_0 = \mathrm{Id}$, $\omega_H = d\eta$, $\owedge : (T^*N)^{\otimes 4} \to (\wedge^2 T^*N)^{\otimes 2}$ is the (extended) Kulkarni-Nomizu product and $\varobar : (\wedge^2 T^*N)^{\otimes 2} \to (\wedge^2 T^*N)^{\otimes 2}$ is defined by

$\alpha \otimes \beta \varobar \gamma \otimes \delta := \alpha \otimes \beta \owedge \gamma \otimes \delta + 2\alpha \otimes \beta \otimes \gamma \otimes \delta + 2\gamma \otimes \delta \otimes \alpha \otimes \beta.$

## Specializing to the above series of symmetric spaces

... we prove that $|R^c|^2$ is a non-constant function of $f_Z/f_H$ if $c \neq 0$.

# IX. Explicit form of the 1-loop deformed c-map metric

$$
\begin{aligned}
g^c_{FS} &= \frac{\phi + c}{\phi} g_{\bar{M}} + \frac{1}{4\phi^2}\frac{\phi + 2c}{\phi + c} d\phi^2 \\
&\quad + \frac{1}{4\phi^2}\frac{\phi + c}{\phi + 2c}\left(d\tilde{\phi} + \sum (\zeta^j d\tilde{\zeta}_j - \tilde{\zeta}_j d\zeta^j) + ic(\bar{\partial} - \partial)\mathcal{K}\right)^2 \\
&\quad + \frac{1}{2\phi} g^{pr}_G + \frac{2c}{\phi^2} e^{\mathcal{K}} \left| \sum (X^j d\tilde{\zeta}_j + F_j(X) d\zeta^j)\right|^2,
\end{aligned}
$$

where $(\phi, \tilde{\phi}, \zeta^0, \ldots, \zeta^n, \tilde{\zeta}_0, \ldots, \tilde{\zeta}_n) : G \to \mathbb{R}^{>0} \times \mathbb{R}^{2n+3} \cong \mathbb{R}^{2n+4}$ is a global coord. system on $G$, $c \in \mathbb{R}$, $X^j = z^j/z^0$, $\mathcal{K} = -\log\left(\sum X^i N_{ij} \bar{X}^j\right)$, $N_{ij} = 2\mathrm{Im} F_{ij}$ and

$$
\begin{aligned}
g^{pr}_G &= \sum \mathcal{I}_{ij}(p) d\zeta^i d\zeta^j \\
&\quad + \sum \mathcal{I}^{ij}(p) \left(d\tilde{\zeta}_i + \sum \mathcal{R}_{ik}(p) d\zeta^k\right)\left(d\tilde{\zeta}_j + \sum \mathcal{R}_{j\ell}(p) d\zeta^\ell\right),
\end{aligned}
$$

where $\mathcal{R}_{ij}$, $\mathcal{I}_{ij}$ are real and imaginary parts of

$$
\bar{F}_{ij} + \sqrt{-1}\frac{\sum N_{ik} z^k \sum N_{j\ell} z^\ell}{\sum N_{k\ell} z^k z^\ell}.
$$