Christoph Böhm

Homogeneous Einstein metrics on Euclidean spaces are Einstein solvmanifolds

– Joint work with R. Lafuente –

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Let (M^n, g) be a simply connected, *homogeneous Riemannian manifold*. Then, topologically M^n falls into one of the following three classes:

(A) M^n is non-compact and non-contractible
(B) $M^n \cong \mathbb{R}^n$
(C) M^n is compact

Definition. (M^n, g) is called *Einstein* iff $\mathsf{ric}(g) = \lambda \cdot g$.

Here $\mathsf{ric}(g)$ denotes the Ricci tensor of g and $\lambda \in \mathbb{R}$ the *Einstein constant*. When $\lambda = 0$, we say that (M^n, g) is *Ricci flat*.

Conjecture. (**Alekseevsky, 1975**) Within the class (A) there exist no homogeneous Einstein metrics. The only Einstein metrics within the class (B) are Einstein solvmanifolds.

In this talk (a simply-connected) **solvmanifold** is a pair (S, g), where S is a solvable Lie group endowed with a left-invariant metric g.

Some authors call the first part of this conjecture (conjecture (A)) the Alekseevsky conjecture, and the second part (conjecture (B)) the generalized Alekseevsky conjecture.

Remark. [Alekseevskii 20] Conjecture (A) was stated in [Alekseevskii 75] by D. Alekseevsky. However, already after [Alekseevskii 70] it seemed possible that conjecture (A) might be true. Notice that a non-compact homogeneous space G/H is diffeomorphic to $\mathbb{R}^n$ if and only if H is a maximal compact subgroup of a non-compact Lie group G. Besse writes in 7.58: "This seems quite optimistic since the corresponding statement is false under the weaker assumption of negative Ricci curvature".

Conjecture (B) was not conjectured by D. Alekseevsky. But shortly after 1975 it was noticed, that if one would know that a non-compact homogeneous Einstein space G/H must have H maximal compact, then Conjecture (B) would follow: see Besse 7.58.

There were essentially three major sources for this conjecture: Firstly, a very nice paper by Hano [Hano 57] on homogeneous Kähler spaces G/H, G unimodular and non-compact, stating that if such a space has a non-degenerate Ricci tensor, then it must be a symmetric space. Secondly, from a paper by [Gindikin-Sapiro-Vinberg 63] it follows that a non-compact homogeneous Kähler manifold is a solvmanifold. Finally, and maybe most importantly, constructing homogeneous quaternionic spaces G/H failed, when H is not a maximal compact subgroup: see [Alekseevskii 75]. Then the hope was, that what was true in the Kähler and quaternionic Kähler case should also be true for arbitrary homogeneous Einstein manifolds.

Thm. A (**B.-Lafuente, 2018**) *Homogeneous Einstein metrics on $\mathbb{R}^n$ are isometric to Einstein solvmanifolds.*

Thm. A was known for Ricci-flat homogeneous spaces ([Alekseevskii-Kimel'fel'd 75]), for homogeneous $\mathbb{R}$-bundles over irreducible Hermitian symmetric spaces ([Bérard-Bergery 78]) and in dimensions $n \leqslant 5$ and $n = 7$ ([Arroyo-Lafuente 17]); see [Arroyo-Lafuente 17] for further citations on previous work.

First *unknown case* appears in dimension $n = 6$:

$$\mathbb{R}^6 \cong \widetilde{\mathsf{Sl}(2,\mathbb{R})} \times \widetilde{\mathsf{Sl}(2,\mathbb{R})} .$$

The difficulty in proving **Thm. A** is that $\mathbb{R}^n$ admits *many presentations* as a homogeneous space, for instance:

- $\mathbb{R}^n \cong$ *solvmanifold* S;
- $\mathbb{R}^{3k} \cong (\widetilde{\mathsf{Sl}(2,\mathbb{R})})^k, \ k \geqslant 1$;
- $\mathbb{R}^{2m+1} \cong \mathsf{SO}(m,\widetilde{2)/\mathsf{SO}}(m), \ m \geqslant 3$.

There exist further presentation of $\mathbb{R}^n = G/H$, with G semisimple, which all are related to non-compact, hermitian symmetric spaces. There are also (semi-direrct) products (with non-product) metrics of these two classes of examples.

Thm. (**Lauret, 2010**) *Einstein solvmanifolds are* **standard**.

Definition. Let (S, g) be a solvmanifold and denote by

$$\mathfrak{s} = T_e S = \mathsf{Lie}(S)$$

the Lie algebra of S. Consider the g-orthogonal decomposition

$$\mathfrak{s} = \mathfrak{n} + \mathfrak{a} , \quad \mathfrak{n} = \text{nilradical of } \mathfrak{s} , \quad \mathfrak{a} = \mathfrak{n}^{\perp_g} .$$

Then, (S, g) is called *standard* if $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{s}$.

The notion of standard solvmanifolds was introduced by J. Heber ([Heber 98]) who also obtained:

(i) many deep structure results for Einstein solvmanifolds;
(ii) finiteness results for the modified Ricci tensor $\mathsf{ric}_\star(g)$;
(iii) GIT appearing.

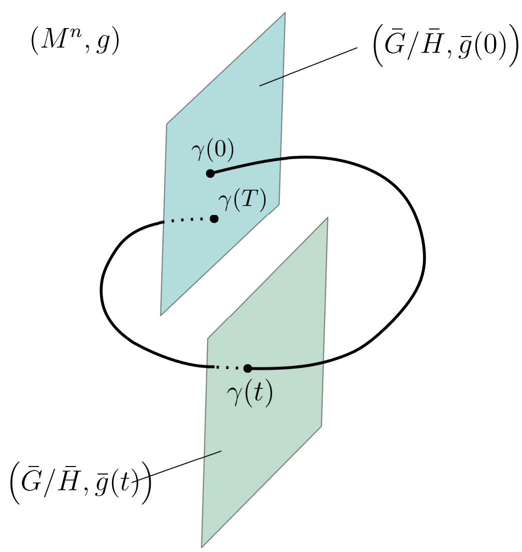
We now state a theorem which at first glance has no relation to **Thm. A**.

Thm. B (**B.-Lafuente, 2018**) *Suppose that (M^n, g) admits an effective, cohomogeneity-one action of a Lie group $\bar{G}$ with closed, integrally minimal orbits and $M/\bar{G} \cong \mathsf{S}^1$. If in addition (M^n, g) is orbit-Einstein with negative Einstein constant, then all orbits are standard homogeneous spaces.*

Let $p \in M^n$ and $\gamma(t)$ be a normal (unit-speed) geodesic, intersecting all orbits perpendicularly. Then

$$g = dt^2 + \bar{g}(t)\,,$$

where $(\bar{g}(t))_{t\in\mathbb{R}}$ is a family of homogeneous metrics on $\bar{G}/\bar{H}$. Notice, that since γ does not need to be periodic, even though $M/\bar{G} \cong \mathsf{S}^1$, the curve $\bar{g}(t)$ needs not be periodic either!



Definition. (M^n, g) is called integrally minimal, if

$$\int_0^T \operatorname{tr} \bar{L}(t)\, dt = 0\,,$$

where $\operatorname{tr} \bar{L}(t)$ = *mean curvature* of $(\bar{G}/\bar{H}, \bar{g}(t))$ and $T > 0$ is the period of $\gamma(t)$, i.e. $\gamma(0)$ and $\gamma(T)$ lie in the same $\bar{G}$-orbit.

Notice, that if all orbits are minimal, then (M^n, g) is of course also integrally minimal.

Definition. A cohomogeneity-one manifold (M^n, g) is called *orbit-Einstein* with Einstein constant $\lambda = -1$, if

$$\mathsf{ric}(g)(X, X) = -g(X, X)$$

for all vectors X *tangent to orbits*.

The orbit-Einstein equation with Einstein constant $\lambda \equiv -1$ can be considered as a *second order Ricci flow* on the space $\mathcal{M}^{\bar{G}}$ of invariant metrics on $\bar{G}/\bar{H}$ given by:

$$\tfrac{D^2}{dt^2}\bar{g}_t = -(\operatorname{tr} \bar{L}_t) \cdot \bar{g}_t' + 2\,\mathsf{ric}(\bar{g}_t) + 2\bar{g}_t\,.$$

Definition. We call the homogeneous space $(\bar{G}/\bar{H}, \bar{g})$ *standard*, if the following is true: considering the nilradical $\bar{N}$ of $\bar{G}$, i.e. the connected Lie subgroup of $\bar{G}$ corresponding to $\bar{\mathfrak{n}}$. Assume for simplicity, that $\bar{N}$ acts freely on $\bar{G}/\bar{H}$. Then $(\bar{G}/\bar{H}, \bar{g})$ *is called standard* if the horizontal distribution of the Riemannian submersion

$$\bar{N} \to \bar{G}/\bar{H} \xrightarrow{\pi} (\bar{G}/\bar{H})/\bar{N}$$

is integrable.

Methods to prove **Thm. B**:

- ODE's methods for the orbit-Einstein equation, for instance we derive a maximum principle for

$$h(t) := \tfrac{1}{2} \cdot \big(\mathsf{scal}_\star(\bar{g}(t)) - \mathsf{scal}(\bar{g}(t))\big) \geqslant 0$$

where $\mathsf{scal}_\star(\bar{g}) = \operatorname{tr} \mathsf{ric}_\star(\bar{g})$ denotes the trace of modified Ricci tensor. Notice, that algebraically, $h = 0$ iff $\bar{G}$ is *unimodular*.

- In order to show such a maximum principle, we use curvature estimates coming from GIT for

$$\begin{aligned} \mathsf{Ric}(\bar{g}) &= \mathsf{Ric}_\star(\bar{g}) - S_{\bar{g}}(\operatorname{ad}(H_{\bar{g}})) \\ &= M(\bar{g}) - \tfrac{1}{2} \cdot B(\bar{g}) - S_{\bar{g}}(\operatorname{ad}(H_{\bar{g}})) \end{aligned}$$

Here $\mathsf{Ric}(\bar{g})$ denotes the Ricci-endomorphism, corresponding to $\mathsf{ric}(\bar{g})$ and $2h(\bar{g}) = \bar{g}(H_{\bar{g}}, H_{\bar{g}})$.

We will now describe the curvature estimates in greater detail which is due to Jorge Lauret ([Lauret 10]) who used it to show standardness of Einstein solvmanifolds. See e.g. [Böhm-Lafuente 18] and the preprint [Böhm-Lafuente 17] for further details.

We consider the simplest case $M^m = G$, i.e. the isotropy group H is trivial. As above we denote by

$$\mathcal{M}^G = \mathsf{Sym}_+(\mathbb{R}^m)$$

the space of left-invariant metrics on G and by

$$\mathfrak{g} = T_e G = \mathbb{R}^m$$

the Lie algebra of G. Furthermore we set

$$\mu_0 \equiv \mu_{\mathfrak{g}} = [\ ,\ ] = \text{Lie bracket of Killing vector fields on } \mathfrak{g}\,.$$

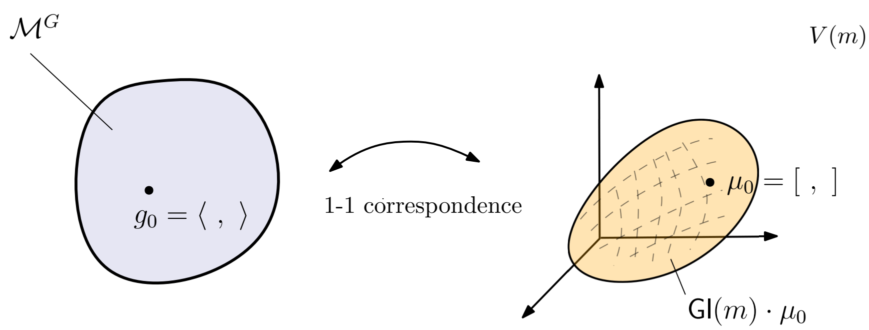
We also fix a scalar product $g_0 = \langle\ ,\rangle \in \mathsf{Sym}_+(\mathbb{R}^m)$. Let us denote by

$$V(m) = \Lambda^2\,\mathbb{R}^m \otimes \mathbb{R}^m\,,$$

the space of brackets, where g_0 is a left-invariant inner product of $\mathfrak{g}$ allowing us to identity $\mathbb{R}^n$ with its dual space. Notice that $\mu_0 \equiv \mu_{\mathfrak{g}} \in V(m)$.

The space $V(m)$ comes with a natural $\mathsf{GL}(m,\mathbb{R})$-action given by

$$(h,\mu) \mapsto h\mu(h^{-1}\cdot, h^{-1}\cdot)$$



The key observation is that there is an essentially one-to-one map between $\mathcal{M}^G$ and the $\mathsf{GL}(m)$-orbit $\mathsf{GL}(m).\mu_{\mathfrak{g}}$; in fact only $\mathsf{Aut}(G)$-orbits of metrics and $\mathsf{O}(m)$-orbits of brackets correspond to each other, which is good enough for applications.

Now, the space $V(m)\backslash\{0\}$ admits a Kirvan-Ness-stratification, that is

$$V(m)\backslash\{0\} = \bigcup_{\beta\in B} S_\beta\,,$$

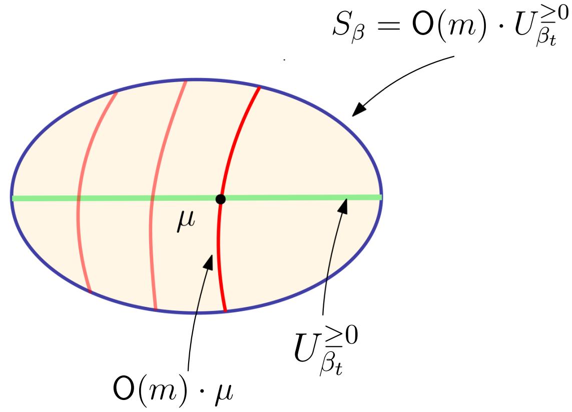
where B is a finite subset of $\mathsf{Sym}(\mathbb{R}^m)$ and S_β are smooth, disjoint submanifolds, which are $\mathsf{GL}(m)$-invariant. In particular, the orbit $\mathsf{GL}(m).\mu_{\mathfrak{g}}$ is contained in one stratum S_β.

This allows us to associate to the Lie group G a stratum label β!

Each stratum S_β has a slice $U^{\geqslant 0}_{\beta_+}$ with the following property: $S_\beta = \mathrm{O}(m)\cdot U^{\geqslant 0}_{\beta_+}$.

Moreover, for brackets $\mu \in U^{\geqslant 0}_{\beta_+}$ we have

$$\langle M(\mu), \beta_+\rangle = \mathrm{tr}\big(M(\mu)\cdot\beta_+\big) \geqslant 0$$



where

$$\beta_+ := \frac{\beta}{\|\beta\|^2} + \mathrm{Id}_m$$

is positive semidefinite. Now, it is not difficult to see that this implies

$$\langle \mathsf{Ric}_\star(\mu), \beta_+ \rangle \geqslant 0$$

Notice, that if G is semisimple, then $\beta_+ = 0$, that is the above estimate gives no information whatsoever. Therefore, we consider cohomogeneity-one actions on homogeneous spaces which are algebraically better in the sense that the corresponding β_+ never vanish.

We explain next, why we can apply **Thm. B** to homogeneous metrics of $\mathbb{R}^n$.

We have

$$\begin{aligned} G &= \mathsf{Sl}(2,\mathbb{R}) \times \mathsf{Sl}(2,\mathbb{R}) \\ &= K \cdot A \cdot N \times \mathsf{Sl}(2,\mathbb{R}) \\ &= K \cdot \bar{G} \end{aligned}$$

where

$$\bar{G} = A \cdot N \times \mathsf{Sl}(2,\mathbb{R}) \quad \text{and} \quad K = \mathsf{SO}(2) = \mathsf{S}^1 .$$

Notice that $\bar{G}$ acts on $(M^6 = \mathsf{Sl}(2,\mathbb{R})^2, g)$ with cohomogeneity one and orbit space $K = \mathsf{S}^1$. Similarly, for any homogeneous Euclidean space $(\mathbb{R}^n, g)$ such a nonunimodular Lie group $\bar{G}$ can be found. This is why we can apply **Thm. B** to prove **Thm. A**.

Finally, let us indicate how to prove **Thm. A** after applying **Thm. B**. We know that

- $\mathbb{R}^n = G/H$ is standard by (**Lafuente - Lauret 2014**)
- $(\bar{G}/\bar{H}, \bar{g}_t)$ is standard for all t by **Thm. B**
- We use several $\bar{G}$'s, and a *Levi-decomposition*

$$G = L \ltimes R .$$

where L is a (maximal) semisimple and R is a (maximal) solvable, normal connected Lie group with $H \subset L$; this is possible by Lafuente-Lauret (2014).

- Then, it follows that $(L/H, g|_{L/H})$ is awesome, i.e. it admits an *orthogonal Cartan decomposition*.
- This gives a contradiction unless, $L = \{e\}$ (**Nikonorov 2000**), showing the claim.

We finally address the question, whether the methods indicated above can be generalized?

We consider the case $G = \mathsf{Sl}(m, \mathbb{R})$, $m \geqslant 3$. We have

$$G \;=\; \mathsf{SO}(m) \cdot S = K \cdot A \cdot N\,,$$

where

$$K = \mathsf{SO}(m) \text{ and } S := A \cdot N$$

For any left-invariant metric g on $\mathsf{Sl}(m, \mathbb{R})$ we obtain a Riemannian submersion

$$\begin{array}{ccc} S = AN & \longrightarrow & (\mathsf{Sl}(m,\mathbb{R}), g) \\ & & \big\downarrow \pi \\ & & (B = K, \bar{g}) \end{array}$$

Let us denote by $\mathscr{V}$ = vertical distribution associated to the fibers and by $\mathscr{H}$ = horizontal distribution = $\mathscr{V}^{\perp}$.
Then, the *Einstein equation* with Einstein constant $\lambda = -1$, reduces to

(E1) $\mathsf{ric}(g)|_{\mathscr{V}\times\mathscr{V}} = -g|_{\mathscr{V}\times\mathscr{V}}$,
(E2) $\mathsf{ric}(g)|_{\mathscr{V}\times\mathscr{H}} = 0$,
(E3) $\mathsf{ric}(g)|_{\mathscr{H}\times\mathscr{H}} = -g|_{\mathscr{H}\times\mathscr{H}}$.

Compared to the above cohomogeneity one situation the following serious problems occur now:

1. It is expected, that it is not enough to use only (E1).
2. $\mathscr{H}$ might not be integrable, that is the corresponding A-tensor might not vanish.
3. The base manifold $(B = K, \bar{g})$ will in general not be Ricci-flat; the Ricci curvature of that space appears as a term in (E3).

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